



Eversible Rings and Zero-Divisors

by

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Abstract

This thesis centers on the study of eversible rings, a class of rings in which every one-sided zero-divisor is necessarily a two-sided zero-divisor. The concept of eversibility generalizes the idea of reversibility and offers a new perspective on the structure of noncommutative rings. To motivate this study, Chapter 1 provides a historical and conceptual overview of zero-divisors and their significance in both commutative and noncommutative settings. Chapter 2 introduces some standard rings such as directly finite rings, von Neumann regular rings, trivial extensions and skew polynomial rings, which lay the groundwork for the main investigation. In Chapter 3, the focus shifts to the study of zero-divisors and eversibility in specific ring contexts, including formal triangular matrix rings, upper triangular matrix rings, polynomial rings and formal power series rings. The final section of Chapter 3 critically addresses several incorrect results from previous studies on eversibility. Through carefully constructed counterexamples, the thesis disproves these erroneous claims, further refining our understanding of the conditions under which eversibility holds in these contexts.

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The inspiration for this thesis stems from a result related to eversible rings, initially published by Ebrahim Ghashghaei and his partners. Although their result was later found to be not fully correct, it served as a starting point for my research. Dr. Zhou and I have worked together to correct the errors, which was an invaluable learning experience.

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Statement of contribution

Chapters 2 and 3 of this thesis are based on the paper “Zero-divisors and eversible rings: Examples and Counter-examples” by Ranran Zhao and Yiqiang Zhou, published in the Journal of Algebra and Its Applications.

The research presented in that paper was carried out by the author under the supervision of Professor Yiqiang Zhou at Memorial University of Newfoundland.

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List of symbols

\mathbb{C}	the set of complex numbers
\mathbb{R}	the set of real numbers
\mathbb{Q}	the set of rational numbers
\mathbb{Z}	the set of integers
\mathbb{Z}_n	the quotient ring $\mathbb{Z}/n\mathbb{Z}$
$\mathbb{M}_{m,n}(R)$	the ring of $m \times n$ matrices over R
$\mathbb{M}_n(R)$	the ring of $n \times n$ matrices over R
$\mathbb{T}_n(R)$	the ring of $n \times n$ upper triangular matrices over R
R_n	the set of all $1 \times n$ matrices over R
R^n	the set of all $n \times 1$ matrices over R
$U(R)$	the group of invertible elements (units) in the ring R
$U^*(R)$	the set of all non-units in the ring R
$U_l(R)$	the set of all elements which have a right inverse in the ring R
$U_r(R)$	the set of all elements which have a left inverse in the ring R
$U_l^*(R)$	the set of all elements which have no right inverse in the ring R
$U_r^*(R)$	the set of all elements which have no left inverse in the ring R
$\mathbf{l}_R(a)$	the set of left annihilators of a in R
$\mathbf{r}_R(a)$	the set of right annihilators of a in R
$zd(R)$	the set of two-sided zero-divisors in R
$zd^*(R)$	the set of non-zero-divisors in R
$zd_l(R)$	the set of left zero-divisors in R
$zd_l^*(R)$	the set of left non-zero-divisors in R
$zd_r(R)$	the set of right zero-divisors in R
$zd_r^*(R)$	the set of right non-zero-divisors in R

Chapter 1

Introduction - The Evolution of Zero-Divisor Theory

Throughout this chapter, all rings are assumed to be associative and unital.

1.1 Origins of Zero-Divisors

The concept of zero-divisors, though not explicitly formulated in early ring theory, emerged naturally from the study of commutative algebra in the 19th century, particularly in efforts to generalize the arithmetic of integers. Foundational work by mathematicians such as Dedekind and Kronecker on ideal theory laid the structural groundwork for understanding rings in which multiplication could fail to preserve nonzero elements. In particular, Dedekind's development of the theory of algebraic integers introduced the class of rings now known as Dedekind domains (i.e. integral domains in which every nonzero ideal factors uniquely into prime ideals [5]). The absence of zero-divisors in these domains was not the focus of his work, but it played a crucial enabling role in ensuring the desirable ideal-theoretic properties that underpinned algebraic number theory.

The formalization of abstract ring theory advanced significantly in the early 20th century, particularly through the contributions of Emmy Noether and Emil Artin. In her landmark paper *Idealtheorie in Ringbereichen* [14], Noether introduced a systematic framework for understanding ring structures through the lens of ideals. While

zero-divisors were not a focus of the paper, the theory she developed, especially concepts such as primary decomposition and the ascending/descending chain conditions, would later become essential tools for analyzing how zero-divisors affect ring decompositions, particularly in Artinian and Noetherian settings.

Artin, together with Nesbitt and Thrall, extended these structural insights to non-commutative rings in their treatise *Rings with Minimum Condition* [2], establishing key properties of Artinian rings and their modules. Although zero-divisors are not explicitly central in their exposition, modern interpretations of their results, particularly in the context of matrix rings and semisimple modules, reveal that the presence of zero-divisors is deeply tied to the failure of injectivity in module homomorphisms and the reducibility of representations. These perspectives illustrate how early structural ring theory laid a foundation for the later systematic study of zero-divisors in both commutative and noncommutative contexts.

1.2 Zero-Divisors in Noncommutative Rings

Zero-divisors can signal structural intricacies within a ring. In local rings, elements outside the maximal ideal are units, while those inside often include zero-divisors. As described in [3, Chapters 1, 2, 3], the nilradical and Jacobson radical capture the behavior of such elements, both of which play key roles in localization and completion [3, Chapters 4, 10]. The interaction between radicals and zero-divisors thus provides insight into the internal structure of local rings and their deformations.

While idempotent elements ($e^2 = e$) and units (invertible elements) play significant roles in ring decomposition, their definitions can be restrictive in certain contexts. In particular, as noted in [10], non-trivial idempotents enable decompositions of commutative rings into direct sums of ideals, revealing internal structure through orthogonal splitting. Furthermore, any non-trivial idempotent satisfies $e(1-e) = 0$, which implies that e and $1-e$ are zero-divisors. This identity highlights a structural link between idempotents and zero-divisors, suggesting that the presence of nontrivial idempotents necessarily introduces zero-divisors into the ring.

In noncommutative ring theory, the interplay between idempotents and structural decompositions becomes particularly significant in the study of semiperfect rings. According to Lam [12], a ring R is semiperfect if $R/\text{rad } R$ is semisimple, and every

idempotents in $R/\text{rad } R$ can be lifted to R . These conditions enable a canonical decomposition of R into a finite direct sum of indecomposable projective modules, each generated by a primitive idempotent. Although zero-divisors are not the primary focus in this context, the existence of nontrivial idempotents (which satisfy $e(1 - e) = 0$) naturally introduces zero-divisors into the ring. Thus, while not explicitly emphasized, the presence of zero-divisors is structurally entangled with the decomposition behavior of semiperfect rings through the action of idempotents.

The transition to noncommutative ring theory introduced complexities in the behavior of zero-divisors. Nathan Jacobson, in his seminal work *Structure of Rings* [7], defined the Jacobson radical as the intersection of all maximal right ideals, characterizing it as the set of elements that annihilate all simple right modules. While not all elements in the Jacobson radical are zero-divisors, their annihilating property connects them to the broader study of ring structure and decomposition.

1.3 Annihilators: From Tools to Theory

The interplay between ring theory and linear algebra provides a fruitful framework for understanding zero-divisors. In particular, the **annihilator** of an element $a \in R$, defined as $\text{Ann}_R(a) = \{b \in R \mid ab = 0\}$, naturally corresponds to the kernel of the left multiplication map $\varphi_a : R \rightarrow R$, where $\varphi_a(b) = ab$. This correspondence suggests a linear-algebraic perspective: elements with trivial annihilators (i.e., non-zero-divisors) behave analogously to injective endomorphisms, preserving structural information in a manner reminiscent of injective linear transformations. While this viewpoint is not explicitly developed in [13], McCoy's formulation of zero-divisors in polynomial rings motivates such analogies and highlights the potential for cross-pollination between linear and ring-theoretic techniques.

The algebraic significance of annihilators became more pronounced with McCoy's seminal 1942 result on zero-divisors in polynomial rings [13]. Specifically, for a commutative ring R , a polynomial $f(x) \in R[x]$ is a zero-divisor if and only if there exists a nonzero element $c \in R$ such that $f(x)c = 0$. This theorem, which characterizes zero-divisors via annihilation in the base ring, underscores the role of annihilators as primary indicators of multiplicative failure. However, the restriction to commutative rings is essential: attempts to extend the result to noncommutative settings encounter

counterexamples, such as the one presented by Jones and Weiner in [9], illustrating the asymmetry and complexity introduced by noncommutativity.

1.4 The Rise of Eversible Rings

The study of symmetric behavior among zero-divisors has been notably explored through the concept of **reversible rings**, where the condition $ab = 0$ implies $ba = 0$. Although the notion predates his work, Cohn [4] provided significant structural analysis of reversible rings, particularly in the context of matrix algebras and noncommutative settings. Subsequent work by Anderson and Valdes-Leon [1] further investigated commutative rings with zero-divisors, emphasizing the role of symmetric annihilation in factorization theory.

The notion of **eversible rings** was introduced in [6] as a generalization of reversibility. A ring R is said to be eversible [6] if every one-sided zero-divisor is also a two-sided zero-divisor. Unlike reversible rings, eversible rings do not require that the left and right annihilators coincide. That is, while $a \in R$ may have both a left and a right annihilator, these need not be equal, and the implication $ab = 0 \Rightarrow ba = 0$ may fail.

Chapter 2

Structural Tools for Eversible Constructions

Throughout this chapter, all rings are assumed to be associative and unital. The goal is to collect several classical structures and technical lemmas that will serve as key ingredients in the construction and analysis of eversible rings in Chapter 3. While not all materials directly involve eversibility, their structural properties will be essential in shaping the examples and counterexamples developed later.

Definition 2.0.1. Let R be a ring and $a \in R$. We say $x \in R$ is a **left annihilator** of a in R if $xa = 0$. The set of all left annihilators of a in R is denoted by $\mathbf{l}(a)$ or $\mathbf{l}_R(a)$. Similarly, We say $x \in R$ is a **right annihilator** of a in R if $ax = 0$. The set of all right annihilators of a in R is denoted by $\mathbf{r}(a)$ or $\mathbf{r}_R(a)$.

Notation. For convenience, we will define several notations for specific types of zero-divisor in R as below:

$zd_l(R) := \{a \in R \mid \mathbf{r}(a) \neq \{0\}\}$ means all left zero-divisors in R .

$zd_l^*(R) := \{a \in R \mid \mathbf{r}(a) = \{0\}\}$ means all left non-zero-divisors in R .

$zd_r(R) := \{a \in R \mid \mathbf{l}(a) \neq \{0\}\}$ means all right zero-divisors in R .

$zd_r^*(R) := \{a \in R \mid \mathbf{l}(a) = \{0\}\}$ means all right non-zero-divisors in R .

$zd(R) := zd_l(R) \cap zd_r(R)$ means all two-sided zero-divisors in R .

$zd^*(R) := zd_l^*(R) \cap zd_r^*(R)$ means all non-zero-divisors in R .

Notation. For convenience, we will henceforth denote $\mathbf{r}(a) = 0$ to represent $\mathbf{r}(a) = \{0\}$.

Definition 2.0.2. A ring R is called **directly-finite** [6] if every one-sided invertible element is a unit. Equivalently, R is directly-finite if $ab = 1 \iff ba = 1$ for any $a, b \in R$.

Notation. We will write $U(R)$ to denote the group of invertible elements (units) in R .

Let $U_l(R) := \{a \in R \mid aR = R\}$ be the set of elements which have a right inverse.

Let $U_r(R) := \{a \in R \mid Ra = R\}$ be the set of elements which have a left inverse.

We also define $U_l^*(R) := \{a \in R \mid aR \neq R\}$, $U_r^*(R) := \{a \in R \mid Ra \neq R\}$ and $U(R) := U_l^*(R) \cup U_r^*(R)$.

Proposition 2.0.3. *Let R be a ring. The followings are equivalent:*

- (1) R is directly-finite.
- (2) $R = U(R) \cup U_l^*(R)$.
- (3) $R = U(R) \cup U_r^*(R)$.

Proof. We first show (1) \iff (2), then (1) \iff (3) will be similar.

To show (1) \implies (2): Suppose R is directly-finite, i.e. one-sided unit is two-sided, i.e. $U(R) = U_l(R) = U_r(R)$. It follows that $R = U_l(R) \cup U_l^*(R) = U(R) \cup U_l^*(R)$.

To show (1) \impliedby (2): Suppose $R = U(R) \cup U_l^*(R)$.

- If $a \in U_l(R)$, then $a \notin U_l^*(R)$, so $a \in R \setminus U_l^*(R) \subseteq U(R)$. Hence, $U_l(R) = U(R)$.
- If $a \in U_r(R)$, i.e. $b \in R$ such that $ba = 1$, so $b \in U_l(R) \subseteq U(R)$, and so b has both left and right inverse, which gives us $ab = 1$. Hence, $a \in U_l(R) \subseteq U(R)$, and so $U_r(R) = U(R)$.

Thus, R is directly-finite. □

Corollary 2.0.4. *Let R be a ring. The followings are equivalent:*

- (1) R is eversible.

(2) R is directly finite with $U_l^*(R) \subseteq zd(R) \cup zd^*(R)$.

(3) R is directly finite with $U_r^*(R) \subseteq zd(R) \cup zd^*(R)$.

(4) R is directly finite with $U_l^*(R) \cap U_r^*(R) \subseteq zd(R) \cup zd^*(R)$.

Proof. To show (1) \Rightarrow (2): Suppose R is eversible, i.e. $R = zd(R) \cup zd^*(R)$. Then $U_l^*(R) \subseteq R = zd(R) \cup zd^*(R)$. Also, by the proposition 2.13 in [6], eversibility implies directly-finite property.

To show (2) \Rightarrow (3): Suppose R is directly finite with $U_l^*(R) \subseteq zd(R) \cup zd^*(R)$. Since $U(R) = U_l(R) = U_r(R)$, we know $U_l^*(R) = U_r^*(R)$, and so $U_r^*(R) = U_l^*(R) \subseteq zd(R) \cup zd^*(R)$.

To show (3) \Rightarrow (4): Suppose R is directly finite with $U_r^*(R) \subseteq zd(R) \cup zd^*(R)$. Then, $U_l^*(R) \cap U_r^*(R) \subseteq U_r^*(R) \subseteq zd(R) \cup zd^*(R)$.

To show (4) \Rightarrow (1): Suppose R is directly finite with $U_l^*(R) \cap U_r^*(R) \subseteq zd(R) \cup zd^*(R)$. Our goal is to show R is eversible, i.e. $R = zd(R) \cup zd^*(R)$. However, we only need to show $R \subseteq zd(R) \cup zd^*(R)$. Now, let $a \in R$, then there are two cases:

- Case 1: If $a \in U_l(R) \cup U_r(R)$, then $a \in U(R)$ as R is directly-finite. Hence $a \in zd^*(R) \subseteq zd(R) \cup zd^*(R)$.
- Case 2: If $a \notin U_l(R) \cup U_r(R)$, then $a \in U_l^*(R) \cap U_r^*(R)$, hence by (4) we get $a \in zd(R) \cup zd^*(R)$.

□

Definition 2.0.5. A ring is called **von Neumann regular** if for any $a \in R$, $a = aba$ for some $b \in R$.

Definition 2.0.6. A ring is called **π -regular** [6] if for any $a \in R$, $a^n \in a^n R a^n$ for some $n \geq 1$.

Proposition 2.0.7. *If a ring R is von Neumann regular, then R is π -regular.*

Definition 2.0.8. A ring R is called **unit π -regular** if for any $a \in R$, $a^n = a^n u a^n$ for some natural number $n \geq 1$ and a unit $u \in R$.

Definition 2.0.9. A ring R is called **strongly π -regular** if for any $a \in R$, $a^n = eu = ue$ for some natural number $n \geq 1$, a unit $u \in R$ and an idempotent $e \in R$.

Definition 2.0.10. Let R be a ring and σ be a ring endomorphism of R . Let $R[x; \sigma] := \{\sum_{i=0}^n a_i x^i \mid a_i \in R, n \geq 0, xb = \sigma(b)x \ \forall b \in R\}$, which we call the **ring of left skew polynomials**. Elements in $R[x; \sigma]$ are called **left skew polynomials**. [12]

Remark 2.0.11. Similarly, we can define the ring of left skew power series. This definition is developed by relaxing the assumption that “elements of R commute with x ” in the setting of polynomial ring and the power series ring over R .

Definition 2.0.12. For a bimodule M over R , the **trivial extension of R by M** , denoted $R \rtimes M$ is the ring whose underlying additive group is $R \oplus M$ and whose multiplication is given by $(a, x)(b, y) = (ab, ay + xb)$

Notation. Let M be a right R -module and N be a left R -module. If $a \in R, x \in M, y \in N$, then we write the notations for annihilators as below:

$$\begin{aligned} \mathbf{l}_M(a) &:= \{m \in M \mid ma = 0\} \\ \mathbf{r}_N(a) &:= \{n \in N \mid an = 0\} \\ \mathbf{r}_R(x) &:= \{r \in R \mid xr = 0\} \\ \mathbf{l}_R(y) &:= \{r \in R \mid ry = 0\} \end{aligned}$$

Proposition 2.0.13. Let $T = R \rtimes M$ be a trivial extension of a ring R by a bimodule M over R . Let $\alpha = (a, x) \in T$. Then:

$$\begin{aligned} (1) \quad \mathbf{r}_T(\alpha) = 0 & \text{ iff } \left[\mathbf{r}_M(a) = 0 \text{ and } \mathbf{r}_R(x) \cap \mathbf{r}_R(a) = 0 \text{ and } aM \cap x\mathbf{r}_R(a) = 0 \right] \\ (2) \quad \mathbf{l}_T(\alpha) = 0 & \text{ iff } \left[\mathbf{l}_M(a) = 0 \text{ and } \mathbf{l}_R(x) \cap \mathbf{l}_R(a) = 0 \text{ and } Ma \cap \mathbf{l}_R(a)x = 0 \right] \end{aligned}$$

Proof. We will prove (1) first, and the proof of (2) will follow similarly to that of (1).

(1) \Rightarrow : Suppose $\mathbf{r}_T(\alpha) = 0$. Let $x' \in \mathbf{r}_M(a)$ and our goal is to show $x' = 0$. Let $a' \in \mathbf{r}_R(x) \cap \mathbf{r}_R(a)$ and our goal is $a' = 0$. Let $ax'' = xa'' \in aM \cap x\mathbf{r}_R(a)$ and our goal is $ax'' = xa'' = 0$.

Consider $\alpha(0, x') = (a, x)(0, x') = (0, ax') = (0, 0)$ as $x' \in \mathbf{r}_M(a)$. Since $\mathbf{r}_T(\alpha) = 0$, we know $(0, x') = (0, 0)$, so $x' = 0$.

Consider $\alpha(a', 0) = (a, x)(a', 0) = (aa', xa')$. Since $a' \in \mathbf{r}_R(x)$, $xa' = 0$. Since $a' \in \mathbf{r}_R(a)$, $aa' = 0$. Hence, $\alpha(a', 0) = 0$, and so $(a', 0) \in \mathbf{r}_T(\alpha) = 0$, which gives us $a' = 0$.

Consider $\alpha(a'', -x'') = (a, x)(x'', -x'') = (aa'', -ax'' + xa'')$. Since $a'' \in \mathbf{r}_R(a)$, we know $aa'' = 0$. Since $ax'' = xa''$, we know $-ax'' + xa'' = 0$. Hence $\alpha(a'', -x'') = 0$, i.e. $(a'', -x'') \in \mathbf{r}_T(\alpha) = 0$, so $a'' = 0$ and $x'' = 0$. Thus $ax'' = xa'' = 0$.

(1) \Leftarrow : Let $\beta = (a', x') \in \mathbf{r}_T(\alpha)$ and our goal is $a' = x' = 0$. Note that $0 = \alpha\beta = (a, x)(a', x') = (aa', ax' + xa')$, which gives $aa' = 0$ and $ax' + xa' = 0$. Observe that $ax' = x(-a') \in aM \cap x\mathbf{r}_R(a) = 0$. Finally, since $xa' = 0$ and $aa' = 0$, we get $a' \in \mathbf{r}_R(x) \cap \mathbf{r}_R(a) = 0$. Also, since $\mathbf{r}_M(a) = 0$ and $ax' = 0$, we know $x' = 0$.

□

Definition 2.0.14. A ring R is called **right Ore** if for any $a, b \in R$ with $b \in zd^*(R)$, there exist $c, d \in R$ with $d \in zd^*(R)$ such that $ad = bc$.

Proposition 2.0.15. *For every right Ore domain R , there is a unique (up to natural R -isomorphism) division ring D containing R as a subring such that every element of D is of the form rs^{-1} for r in R and s nonzero in R . Such a division ring D is called a **ring of right fractions of R** .*

Chapter 3

Eversible Rings: Constructions, Examples, and New Results

This chapter contains important results related to everisible rings, including various examples and counterexamples.

Definition 3.0.1. A ring R is called **everisible** if $zd_l(R) = zd_r(R)$ (i.e. Every left zero-divisor is also a right zero-divisor and conversely). A ring R is called **left-everisible** if $zd_l(R) \subseteq zd_r(R)$. A ring R is called **right-everisible** if $zd_r(R) \subseteq zd_l(R)$.

Example 3.0.2. The ring $R = \mathbb{T}_2(\mathbb{Z}_4)$ is everisible but not reversible. This is because $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 0$, while the reverse product is nonzero.

Proposition 3.0.3. *Let R be a ring. The followings are equivalent:*

- (1) R is everisible.
- (2) $R = zd(R) \cup zd^*(R)$.
- (3) For any $0 \neq a \in R$, $\mathbf{l}_R(a) = 0 \iff \mathbf{r}_R(a) = 0$.

Proof. (1) \iff (2): Note that R is everisible $\iff zd_l(R) = zd_r(R) \iff$ There is no single-sided zero divisor in $R \iff R = zd(R) \cup zd^*(R)$.

- (1) \iff (3): Note that R is everisible $\iff zd_l(R) = zd_r(R) \iff \forall 0 \neq a \in R$,

$$\mathbf{l}_R(a) = 0 \iff \mathbf{r}_R(a) = 0.$$

□

3.1 Classical Examples

We begin with finite rings, von Neumann regular rings, and some standard constructions such as trivial extensions and skew polynomial rings, which will later serve as a base for various eversible ring examples.

Proposition 3.1.1. *Any finite ring R is eversible.*

Proof. Suppose $0 \neq a \in R$ is a left zero-divisor. Define a map $\varphi_a : R \rightarrow R : r \mapsto ra$. Assume for contradiction that a is not a right zero-divisor, then there is no $0 \neq r \in R$ with $ra = 0$, i.e. $\ker(\varphi_a) = (0)$, hence φ_a is injective. Since R is finite, φ_a is surjective. Thus, there exists $r \in R$ such that $ra = 1$, so $a \in U(R)$, which contradicts that $a \in \text{zd}_l(R)$. Similarly, if $0 \neq a \in R$ is a right zero-divisor, we can show a is also a left zero-divisor. Therefore, R is eversible.

□

Proposition 3.1.2. *Any π -regular, directly finite ring R is eversible.*

Proof. Let $a \in R$ be arbitrary. Suppose $\mathbf{r}(a) = \{0\}$ and we want to show $\mathbf{l}(a) = \{0\}$. Since R is π -regular, there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$, and so $a^n(1 - b a^n) = 0$. Since $\mathbf{r}(a) = \{0\}$, we know $1 - b a^n = 0$, so $b a^n = 1$. Since R is directly-finite, we know $a^n b = 1$. Then, $\mathbf{l}(a) = \{0\}$ because if there is $y \in R$ such that $ya = 0$, then $y = y \cdot 1 = y a^n b = 0$. The converse “ $\mathbf{l}(a) = \{0\}$ implies $\mathbf{r}(a) = \{0\}$ ” holds similarly.

□

Corollary 3.1.3. *Any von Neumann regular, directly finite ring R is eversible.*

Proof. By Proposition 2.0.7, R is π -regular. Then by Proposition 3.1.2, R is eversible.

□

Corollary 3.1.4. *Any unit π -regular ring R is eversible.*

Proof. It suffices to show that R is directly finite by Proposition 3.1.2. To show R is directly finite: Let $a \in R$ with $ab = 1$ for some $b \in R$. The goal is to show that a is a unit in R . Since R is unit π -regular, there exist $n \in \mathbb{N}$ and a unit $u \in R$ such that $a^n = a^n u a^n$. Multiply both sides by b^n from the right, we get $a^n b^n = a^n u a^n b^n$. Note that $ab = 1$, so $a^n b^n = a \cdots a(ab)b \cdots b = 1$. Hence, $1 = a^n u$, so a^n is a unit in R because $a^n = u^{-1}$ is a unit in R . It follows that $a^n u = u a^n = 1$, then $a(a^{n-1}u) = (u a^{n-1})a = 1$, which implies a is a unit in R . Thus, R is directly finite. \square

Proposition 3.1.5. *Let R be a ring with an automorphism σ of R and $n \geq 1$. Then $S := R[t; \sigma]/(t^{n+1})$ is eversible if and only if R is eversible.*

Proof. Let $\alpha := a_0 + a_1 t + \cdots + a_n t^n \in S$.

Claim 1: $\mathbf{r}_S(\alpha) = 0$ implies $\mathbf{r}_R(a_0) = 0$.

Proof of claim 1: Assume for contradiction that $\mathbf{r}_R(a_0) \neq 0$, i.e. $\exists b_0 \neq 0 \in R$ such that $a_0 b_0 = 0$. Consider $\beta := b_0 t^n \in S$. Note that $\beta \neq 0$ because $b_0 \neq 0$. Then $\alpha\beta = (a_0 + a_1 t + \cdots + a_n t^n)(b_0 t^n) = a_0 b_0 t^n = 0$ in S , which contradicts $\mathbf{r}_S(\alpha) = 0$.

Claim 2: $\mathbf{l}_S(\alpha) = 0$ implies $\mathbf{l}_R(a_0) = 0$.

Proof of claim 2: Assume for contradiction that $\mathbf{l}_R(a_0) \neq 0$, i.e. $\exists b_0 \neq 0 \in R$ such that $b_0 a_0 = 0$. Since σ is an automorphism, σ^n is also an automorphism, hence $\sigma^n(b_0) \neq 0$. Consider $\beta := \sigma^n(b_0) t^n \in S$ and clearly $\beta \neq 0$. Then $\beta\alpha = (\sigma^n(b_0) t^n)(a_0 + a_1 t + \cdots + a_n t^n) = \sigma^n(b_0) t^n a_0 = \sigma^n(b_0) \sigma^n(a_0) t^n = \sigma^n(b_0 a_0) t^n = 0$, which contradicts $\mathbf{l}_S(\alpha) = 0$.

Claim 3: $\mathbf{r}_R(a_0) = 0$ implies $\mathbf{l}_S(\alpha) = 0$.

Proof of claim 3: Suppose $\exists \beta = b_0 + b_1 t + \cdots + b_m t^m \in S$ such that $\alpha\beta = 0$, i.e. $\sum_{i,j} a_i t^i b_j t^j = 0$. Note that the only term of degree 0 in $\alpha\beta$ is $a_0 b_0$. Since $\alpha\beta = 0$, $a_0 b_0$ must be 0. Therefore $b_0 = 0$ (recall $\mathbf{r}_R(a_0) = 0$). Note that the term of degree 1 in $\alpha\beta$ is $a_1 t b_0 + a_0 b_1 t = a_0 b_1 t = 0$. Hence, $b_1 = 0$. Repeat the same step from degree 2 to degree $m + n$, we get $b_j = 0$ for all $j = 0, \dots, m$. Therefore, $\beta = 0$.

Claim 4: $\mathbf{l}_R(a_0) = 0$ implies $\mathbf{l}_S(\alpha) = 0$.

Proof of claim 4: Suppose $\exists \beta = b_0 + b_1 t + \cdots + b_m t^m \in S$ such that $\beta\alpha = 0$, i.e. $\sum_{i,j} b_j t^j a_i t^i = \sum_{i,j} b_j \sigma^j(a_i) t^{i+j} = 0$. Note that for each $j = 0, \dots, m$, σ^j is an

automorphism, so for any $b_j \in R$, $\exists b'_j \in R$ such that $\sigma^j(b'_j) = b_j$. Also, $\ker(\sigma^j) = \{0\}$ for any $j = 0, \dots, m$. Note that the only term of degree 0 in $\beta\alpha$ is b_0a_0 , which should be 0. Hence $b_0 = 0$ (recall $\mathbf{1}_R(a_0) = 0$). Note that the only term of degree 1 in $\beta\alpha$ is $b_1\sigma(a_0)t + b_0a_1t = \sigma(b'_1a_0)t + b_0a_1t = \sigma(b'_1a_0)t + 0 \cdot a_1t = \sigma(b'_1a_0)t$, which should be 0. Hence $b'_1a_0 \in \ker(\sigma)$ and so $b'_1 = 0$, which follows that $b_1 = \sigma(b'_1) = 0$. Repeat the same step from degree 2 to degree $m+n$, we get $b_j = 0$ for all $j = 0, \dots, m$. Therefore, $\beta = 0$.

By these four claims above, we know $\left[\mathbf{r}_S(\alpha) = 0 \iff \mathbf{r}_R(a_0) = 0 \right]$ and $\left[\mathbf{1}_S(\alpha) = 0 \text{ implies } \mathbf{1}_R(a_0) = 0 \right]$. It follows that S is eversible iff R is eversible. \square

Remark 3.1.6. From the above proof, we see that the automorphism σ cannot be replaced by an endomorphism. Otherwise, there exists an example such that R is eversible but $R[t; \sigma]/(t^{n+1})$ is not eversible:

Example 3.1.7. Let $R = F[x]$ be a polynomial ring over a field F with variable x . Let $\sigma : R \rightarrow R : f(x) \mapsto f(0)$. Then R is eversible but $S := R[t; \sigma]/(t^2)$ is not eversible.

Proof. Let $\alpha = a + bt \in S$. We will first characterize the one-sided annihilators of α in S :

Claim 1: $\mathbf{1}_S(\alpha) = 0$ implies $\mathbf{1}_R(\sigma(a)) = 0$.

Proof of claim 1: Assume for contradiction that $\mathbf{1}_R(\sigma(a)) \neq 0$, i.e. $\exists 0 \neq c \in R$ such that $c\sigma(a) = 0$. Consider $\beta = ct \in S$ and clearly $\beta \neq 0$. Then $\beta\alpha = (ct)(a+bt) = cta + cbt = c\sigma(a)t = 0$, which contradicts $\mathbf{1}_S(\alpha) = 0$.

Claim 2: $\mathbf{1}_S(\alpha) = 0$ implies $\mathbf{1}_R(a) \cap \mathbf{1}_R(b) = 0$.

Proof of claim 2: Assume for contradiction that $\exists 0 \neq c \in \mathbf{1}_R(a) \cap \mathbf{1}_R(b)$. Then $ca = 0$ and $cb = 0$, so $c\alpha = c(a + bt) = ca + cbt = 0 + 0 \cdot t = 0$, which contradicts $\mathbf{1}_S(\alpha) = 0$.

Claim 3: $\mathbf{1}_S(\alpha) = 0$ implies $\mathbf{1}_R(a)b \cap R\sigma(a) = 0$.

Proof of claim 3: Assume for contradiction that $\exists 0 \neq c \in \mathbf{1}_R(a)b \cap R\sigma(a) = 0$. We know $\exists 0 \neq d \in \mathbf{1}_R(a)$ with $c = db$, and $\exists 0 \neq r \in R$ with $c = r\sigma(a)$. Consider

$\beta := d - rt \in S$ and clearly $\beta \neq 0$. Then $\beta\alpha = (d - rt)(a + bt) = da + dbt - r\sigma(a)t - r\sigma(b)t^2 = 0 + ct - ct - 0 = 0$, which contradicts $\mathbf{l}_S(\alpha) = 0$.

Claim 4: If $\mathbf{l}_R(\sigma(a)) = 0$ and $\mathbf{l}_R(a) \cap \mathbf{l}_R(b) = 0$ and $\mathbf{l}_R(a)b \cap R\sigma(a) = 0$, then $\mathbf{l}_S(\alpha) = 0$.

Proof of claim 4: Suppose we have $\beta = c + dt \in S$ satisfying $\beta\alpha = 0$. Our goal is to show $c = 0$ and $d = 0$. Note that $0 = \beta\alpha = (c + dt)(a + bt) = ca + cbt + d\sigma(a)t + d\sigma(b)t^2 = ca + (cb + d\sigma(a))t$, so $ca = 0$ and $cb = -d\sigma(a)$. Recall $\mathbf{l}_R(a)b \cap R\sigma(a) = 0$, which implies "if $\exists x = lb = r\sigma(a)$ for some $l \in \mathbf{l}_R(a)$ and $r \in R$, then $x = 0$ ". Hence $cb = -d\sigma(a) = 0$. Finally, we obtain $d = 0$ from the conditions $d\sigma(a) = 0$ and $\mathbf{l}_R(\sigma(a)) = 0$. Additionally, we have $c = 0$ since $c \in \mathbf{l}_R(a) \cap \mathbf{l}_R(b) = 0$.

Claim 5: $\mathbf{r}_S(\alpha) = 0$ iff $\mathbf{r}_R(a) = 0$.

Proof of claim 5: This proof directly follows from the proofs of Claim 1 and Claim 3 in the context of proposition 3.1.5, as the arguments remain valid even when σ is only an endomorphism.

Based on above Claims 1 to 4, we conclude $\mathbf{l}_S(\alpha) = 0 \iff \left[\mathbf{l}_R(\sigma(a)) = 0, \mathbf{l}_R(a) \cap \mathbf{l}_R(b) = 0 \text{ and } \mathbf{l}_R(a)b \cap R\sigma(a) = 0 \right]$.

We know R is a domain, so R is eversible. However, $S = R[t; \sigma]/(t^2)$ is not eversible in general. Consider $a = x$ and $b = 0$, i.e., $\alpha = x$, so $\mathbf{r}_R(a) = \mathbf{r}_R(x) = 0$, then $\mathbf{r}_S(\alpha) = 0$ by Claim 5. However, note that $\mathbf{l}_R(\sigma(a)) = \mathbf{l}_R(0) \neq 0$, then $\mathbf{l}_S(\alpha) \neq 0$ by Claim 1 to 4. Thus, S is not eversible.

□

Proposition 3.1.8. *Let R be a domain and M a non-trivial bimodule over R . Then $T := R \rtimes M$ is eversible if and only if for any $0 \neq a \in R$, $r_M(a) = 0 \iff l_M(a) = 0$.*

Proof. Note that T is eversible iff $\left[\mathbf{r}_T(\alpha) = 0 \iff \mathbf{l}_T(\alpha) = 0 \forall \alpha \in T \right]$ iff $\left[\mathbf{r}_M(a) = 0 \iff \mathbf{l}_M(a) = 0 \forall a \in R \right]$, where the second "iff" follows by proposition 2.0.13 (recall R is a domain here and so for any $0 \neq a \in R$, $\mathbf{r}_R(a) = \mathbf{l}_R(a) = 0$).

□

Remark 3.1.9. Based on this characterization of eversibility for trivial extensions, we provide several examples of eversible rings:

Example 3.1.10. Let R be a domain. If M is a sub-bimodule of the (R, R) -bimodule $R[[t, t^{-1}]]$, then $R \rtimes M$ is eversible.

Proof. Note that $M = R[[t]]$ or $M = R[t]$ or M is an ideal of R , then we know the way R acts on M from both sides is just the multiplication in $R[[t, t^{-1}]]$. Let $p(t) = \sum_{i \in \mathbb{Z}} a_i t^i \in M$. Fix $0 \neq a \in R$. Then $p(t) \in \mathbf{r}_M(a) \iff a \cdot p(t) = 0 \iff \sum_{i \in \mathbb{Z}} aa_i t^i = 0 \iff aa_i = 0 \forall i \in \mathbb{Z} \iff a_i = 0 \forall i \in \mathbb{Z} \iff p(t) = 0$. Hence, $\mathbf{r}_M(a) = 0$. Similarly, $\mathbf{l}_M(a)$. Therefore, $\mathbf{r}_M(a) = 0 \iff \mathbf{l}_M(a)$ holds for any $0 \neq a \in R$. By proposition 3.1.8, we know $R \rtimes M$ is eversible. □

Example 3.1.11. Let R be a domain and σ a non-injective endomorphism of R . Let $R[[t; \sigma]]$ be the left skew power series ring. Then $T := R \rtimes R[[t; \sigma]]$ is not eversible.

Proof. Since σ is not injective, we can choose $0 \neq a \in R$ with $\sigma(a) = 0$. We will show $\mathbf{l}_T(\alpha) \neq 0$ but $\mathbf{r}_T(\alpha) = 0$.

Consider $(0, t)\alpha = (0, t)(a, 0) = (0, ta) = (0, \sigma(a)t) = (0, 0)$. Hence, $\mathbf{l}_T(\alpha) \neq 0$.

Suppose $(b, p(t)) \in \mathbf{r}_T(\alpha)$ and we write $p(t) = \sum_{n \geq 0} a_n t^n$. Then $0 = (a, 0)(b, p(t)) = (ab, a \cdot p(t))$. Since $ab = 0$, we obtain $b = 0$ as $a \neq 0$ and R is a domain. Since $0 = a \cdot p(t) = \sum_{n \geq 0} aa_n t^n$, we know $aa_n = 0 \forall n \geq 0$, and we obtain $a_n = 0 \forall n \geq 0$, i.e. $p(t) = 0$. Thus, $\mathbf{r}_T(\alpha) = 0$.

Hence, T is not eversible. □

3.2 Formal Triangular Matrix Rings

In this section, we analyze the structure of zero-divisors in formal triangular matrix rings, drawing on the results of [6], and characterize their eversibility.

In the two propositions below, we first establish the characterization of the right and left zero divisors in a formal triangular matrix ring.

Proposition 3.2.1. Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ where A, B are rings and M is an (A, B) -bimodule. Let $\alpha = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in R$. Then $\mathbf{r}_R(\alpha) \neq 0$ if and only if at least one of the following conditions holds:

- (1) $\mathbf{r}_A(a) \neq 0$,
- (2) $\mathbf{r}_M(a) \neq 0$,
- (3) $\mathbf{r}_B(x) \cap \mathbf{r}_B(b) \neq 0$,
- (4) $aM \cap x\mathbf{r}_B(b) \neq 0$.

Proof. To prove \Rightarrow : Suppose $\mathbf{r}_R(\alpha) \neq 0$. Then $\exists \beta \in R$ with $\alpha\beta = 0$. Write $\beta = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix}$, then $\begin{bmatrix} aa' & ax' + xb' \\ 0 & bb' \end{bmatrix} = 0$. Since $aa' = 0$, $a' \in \mathbf{r}_A(a)$.

If $a' \neq 0$, then (1) holds and we are done.

Hence, we assume $a' = 0$.

If $b' = 0$, then x' should be nonzero since $\beta \neq 0$. Also, $ax' = 0$ implies $0 \neq x' \in \mathbf{r}_M(a)$, so (2) holds and we are done.

Hence, we assume $b' \neq 0$.

If $xb' = 0$, then $0 \neq b' \in \mathbf{r}_M(x) \cup \mathbf{r}_B(b)$, so (3) holds and we are done.

Hence, we assume $xb' \neq 0$, then $ax' = x(-b')$ gives us $0 \neq ax' = x(-b') \in aM \cap x\mathbf{r}_B(b)$, so (4) holds and we are done.

To prove \Leftarrow : Suppose (1) holds, then we can choose $0 \neq a' \in \mathbf{r}_A(a)$. Consider $0 \neq \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} \in R$. Then $\alpha \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ 0 & 0 \end{bmatrix} = 0$. Hence $\mathbf{r}_R(\alpha) \neq 0$.

Suppose (2) holds, then we can choose $0 \neq m' \in \mathbf{r}_M(a)$. Consider $0 \neq \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} \in R$. Then $\alpha \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & am' \\ 0 & 0 \end{bmatrix} = 0$. Hence $\mathbf{r}_R(\alpha) \neq 0$.

Suppose (3) holds, then there exists $0 \neq b \in \mathbf{r}_B(x) \cap \mathbf{r}_B(b)$, i.e. $xb' = 0$ and $bb' = 0$. Consider $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} \in R$. Then $\alpha \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & xb' \\ 0 & bb' \end{bmatrix} = 0$. Hence

$\mathbf{r}_R(\alpha) \neq 0$.

Suppose (4) holds, then there exist $m' \in M$ and $b' \in B$ such that $0 \neq am' = xb'$ and $bb' = 0$. Consider $0 \neq \begin{bmatrix} 0 & m' \\ 0 & -b' \end{bmatrix} \in R$. Then $\alpha \begin{bmatrix} 0 & m' \\ 0 & -b' \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & m' \\ 0 & -b' \end{bmatrix} = \begin{bmatrix} 0 & am' - xb' \\ 0 & -bb' \end{bmatrix} = 0$. Hence $\mathbf{r}_R(\alpha) \neq 0$. □

We also have a symmetric version of the above proposition for the left zero-divisors in R :

Proposition 3.2.2. *Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ where A, B are rings and M is an (A, B) -bimodule. Let $\alpha = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in R$. Then $\mathbf{l}_R(\alpha) \neq 0$ if and only if at least one of the following conditions holds:*

- (1) $\mathbf{l}_B(b) \neq 0$,
- (2) $\mathbf{l}_M(b) \neq 0$,
- (3) $\mathbf{l}_A(x) \cap \mathbf{l}_A(a) \neq 0$,
- (4) $Mb \cap \mathbf{l}_A(a)x \neq 0$.

Proof. The proof will follow similarly to that of Proposition 3.2.1 □

Lemma 3.2.3. *Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, $a \in A$ and $b \in B$. Then we have:*

- (1) *If R is left eversible, then “either $\mathbf{r}_A(a) \neq 0$ or $\mathbf{r}_M(a) \neq 0$ implies $\mathbf{l}_A(a) \neq 0$ ”, and “ $\mathbf{r}_B(b) \neq 0$ implies either $\mathbf{l}_B(b) \neq 0$ or $\mathbf{l}_M(b) \neq 0$ ”.*
- (2) *If R is right eversible, then “either $\mathbf{l}_B(b) \neq 0$ or $\mathbf{l}_M(b) \neq 0$ implies $\mathbf{r}_B(b) \neq 0$ ”, and “ $\mathbf{l}_A(a) \neq 0$ implies either $\mathbf{r}_A(a) \neq 0$ or $\mathbf{r}_M(a) \neq 0$ ”.*
- (3) *If R is left eversible, then A is left eversible. If R is right eversible, then B is right eversible.*
- (4) *If R is eversible, then A is left eversible and B is right eversible.*

Proof. Let $\alpha = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$.

To prove (1): Suppose R is left eversible.

If $\mathbf{r}_A(a) \neq 0$, then there exists $0 \neq a' \in A$ with $aa' = 0$. Then $\alpha \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ 0 & 0 \end{bmatrix} = 0$, hence $0 \neq \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{r}_R(\alpha)$ and so $\mathbf{r}_R(\alpha) \neq 0$.

If $\mathbf{r}_M(a) \neq 0$, then there exists $0 \neq m' \in M$ with $am' = 0$. Then $\alpha \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & am' \\ 0 & 0 \end{bmatrix} = 0$, hence $0 \neq \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} \in \mathbf{r}_R(\alpha)$ and so $\mathbf{r}_R(\alpha) \neq 0$.

Thus, we know either $\mathbf{r}_A(a) \neq 0$ or $\mathbf{r}_M(a) \neq 0$ implies $\mathbf{r}_R(\alpha) \neq 0$. Also, since R is left eversible, i.e. $\mathbf{r}_R(\alpha) \neq 0 \Rightarrow \mathbf{l}_R(\alpha) \neq 0$. Then we can choose $0 \neq \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \in \mathbf{l}_R(\alpha)$.

Note that $0 = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \alpha = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a'a & x' \\ 0 & b' \end{bmatrix}$, so $a'a = 0$ and $x' = 0$ and $b' = 0$. Since $\begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \neq 0$, we know $a' \neq 0$. Thus, $a' \neq 0$ and $a'a = 0$, which implies $\mathbf{l}_A(a) \neq 0$.

If $\mathbf{r}_B(b) \neq 0$, $\exists b' \in B$ with $bb' = 0$, and so $\exists 0 \neq \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix}$ such that $\beta \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & bb' \end{bmatrix} = 0$. Hence $\mathbf{r}_R(\beta) \neq 0$. Since R is left eversible, we know $\mathbf{l}_R(\beta) \neq 0$. Consider $0 \neq \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \in R$, then $0 = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a' & x'b \\ 0 & b'b \end{bmatrix}$, which gives us $a' = 0$, $x'b = 0$ and $b'b = 0$. However, we should require $x' \neq 0$ or $b' \neq 0$ since $\beta \neq 0$. Hence, $\mathbf{r}_B(b) \neq 0$ implies either $\mathbf{l}_B(b) \neq 0$ or $\mathbf{l}_M(b) \neq 0$.

To prove (2): The proof is similar to (1).

To prove (3): From (1), we know that if R is left eversible, then $\mathbf{r}_A(a) \neq 0$ implies $\mathbf{l}_A(a) \neq 0$, i.e. A is left eversible. From (2), we know if R is right eversible, then $\mathbf{l}_B(b) \neq 0$ implies $\mathbf{r}_B(b) \neq 0$, i.e. B is right eversible.

To prove (4): Suppose R is eversible, then R is both left eversible and right eversible. By (3), we know A is left eversible and B is right eversible.

□

Corollary 3.2.4. *Let R be a ring and M an R -bimodule. If $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ is left (resp. right) eversible, then R is left (resp. right) eversible.*

Proof. By Lemma 3.2.5, if $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ is left eversible, then R is left eversible. Also if $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ is right eversible, then R is right eversible.

□

Proposition 3.2.5. *Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ where A, B are eversible rings and $M \neq 0$ is an (A, B) -bimodule. Then R is eversible if and only if for any $a \in A$, $b \in B$ and $x \in M$, the following hold:*

- (1) *If $a \in zd^*(A)$ and $b \in zd^*(B)$, then $\mathbf{r}_M(a) = 0$ and $\mathbf{l}_M(b) = 0$.*
- (2) *If $a \in zd(A)$ and $b \in zd^*(B)$, then $\mathbf{l}(a) \cap \mathbf{l}_A(x) \neq 0$ or $\mathbf{l}(a)x \cap Mb \neq 0$.*
- (3) *If $a \in zd^*(A)$ and $b \in zd(B)$, then $\mathbf{r}(b) \cap \mathbf{r}_B(x) \neq 0$ or $x\mathbf{r}(b) \cap aM \neq 0$.*

Proof. To show the direction (\Rightarrow) : Suppose R is eversible. We have 4 cases to discuss:

Case 1: If $a \in zd(A)$ and $b \in zd(B)$, we have nothing to show.

Case 2: If $a \in zd^*(A)$ and $b \in zd^*(B)$. Let $\alpha = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in R$. Then by Proposition 3.2.2, $\mathbf{l}_R(\alpha) = 0$, then by eversibility of R , we know $\mathbf{r}_R(\alpha) = 0$, and so $\mathbf{r}_M(a) = 0$ by Proposition 3.2.1. Let $\beta = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \in R$. Then by Proposition 3.2.1, $\mathbf{r}_R(\beta) = 0$, then by eversibility of R , we know $\mathbf{l}_R(\beta) = 0$, and so $\mathbf{l}_M(b) = 0$ by Proposition 3.2.2. Hence, (1) is proved.

Case 3: If $a \in zd(A)$ and $b \in zd^*(B)$. Let $\alpha = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in R$. Since $\mathbf{r}_A(a) \neq 0$, we have $\mathbf{r}_R(\alpha) \neq 0$ by Proposition 3.2.1. By eversibility of R , $\mathbf{l}_R(\alpha) \neq 0$, i.e. $\exists 0 \neq \beta = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix}$ such that $\beta\alpha = 0$. Note that $0 = \beta\alpha = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} a'a & a'x + x'b \\ 0 & b'b \end{bmatrix}$,

from which we get $b' = 0$ (since $b \in zd^*(B)$) and $a'x = (-x')b \in \mathbf{l}_A(a)x \cap Mb$. If $\mathbf{l}_A(a)x \cap Mb \neq 0$, then we are done. Now, suppose $\mathbf{l}_A(a)x \cap Mb = 0$, then $a'x = x'b = 0$. By the proof of (1), $b \in zd^*(B)$ implies $\mathbf{l}_M(b) = 0$, and so $x' = 0$. Since $\beta \neq 0$, we must have $a' \neq 0$, so $\mathbf{l}_A(a) \cap \mathbf{l}_A(x) \neq 0$. Hence, (2) is proved.

Case 4: If $a \in zd^*(A)$ and $b \in zd(B)$, the proof is similar to the case 3. Hence (3) is proved.

To show the direction (\Leftarrow): Suppose (1) (2) (3) hold. Let $\alpha = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in R$. To show R is eversible, we need to show " $\mathbf{r}_R(\alpha) = 0 \iff \mathbf{l}_R(\alpha) = 0$ ". We have 4 cases to discuss:

Case 1: If $a \in zd(A)$ and $b \in zd(B)$. Note that $a \in zd(A)$ implies $\mathbf{r}_A(a) \neq 0$, so $\mathbf{r}_R(\alpha) \neq 0$ by Proposition 3.2.1. Note that $b \in zd(B)$ implies $\mathbf{l}_B(b) \neq 0$, so $\mathbf{l}_R(\alpha) \neq 0$ by Proposition 3.2.2. Hence, " $\mathbf{r}_R(\alpha) \neq 0 \iff \mathbf{l}_R(\alpha) \neq 0$ " is vacuously true and we are done.

Case 2: If $a \in zd^*(A)$ and $b \in zd^*(B)$. Note that $a \in zd^*(A)$ implies $\mathbf{r}_M(a) = 0$ by assumption (1), and clearly $\mathbf{r}_A(a) = 0$ and $\mathbf{l}_A(a) = 0$. Note that $b \in zd^*(B)$ implies $\mathbf{r}_M(b) = 0$ by assumption (1), and clearly $\mathbf{r}_B(b) = 0$ and $\mathbf{l}_B(b) = 0$. By Proposition 3.2.1, we know $\mathbf{r}_R(\alpha) = 0$. By Proposition 3.2.2, we know $\mathbf{l}_R(\alpha) = 0$. Hence, " $\mathbf{r}_R(\alpha) \neq 0 \iff \mathbf{l}_R(\alpha) \neq 0$ " is vacuously true and we are done.

Case 3: If $a \in zd(A)$ and $b \in zd^*(B)$. Note that $a \in zd(A)$ implies $\mathbf{r}_A(a) \neq 0$, so $\mathbf{r}_R(\alpha) \neq 0$ by Proposition 3.2.1. Now, we will show $\mathbf{l}_R(\alpha) \neq 0$. By (2), we know $\mathbf{l}_A(a) \cap \mathbf{l}_A(x) \neq 0$ or $\mathbf{l}_A(a)x \cap Mb \neq 0$. If $\mathbf{l}_A(a) \cap \mathbf{l}_A(x) \neq 0$, then $\exists 0 \neq a' \in \mathbf{l}_A(a) \cap \mathbf{l}_A(x) \neq 0$. Then, there is $0 \neq \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_R(\alpha)$ because $\begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} \alpha = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} aa' & a'x \\ 0 & 0 \end{bmatrix} = 0$, which shows $\mathbf{l}_R(\alpha) \neq 0$. If $\mathbf{l}_A(a)x \cap Mb \neq 0$, then $\exists 0 \neq a'x = x'b \in \mathbf{l}_A(a)x \cap Mb$. Notice that $a' \neq 0$ and $x' \neq 0$. Then, there is $0 \neq \begin{bmatrix} a' & -x' \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_R(\alpha)$ because $\begin{bmatrix} a' & -x' \\ 0 & 0 \end{bmatrix} \alpha = \begin{bmatrix} a' & -x' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} a'a & a'x - x'b \\ 0 & 0 \end{bmatrix} = 0$, which shows $\mathbf{l}_R(\alpha) \neq 0$. Hence, " $\mathbf{r}_R(\alpha) \neq 0 \iff \mathbf{l}_R(\alpha) \neq 0$ " is vacuously true and we are done.

Case 4: If $a \in zd^*(A)$ and $b \in zd(B)$. Note that $b \in zd(B)$ implies $\mathbf{l}_B(b) \neq 0$, so $\mathbf{l}_R(\alpha) \neq 0$ by Proposition 3.2.2. Now, we will show $\mathbf{r}_R(\alpha) \neq 0$. By (3), we know

$\mathbf{r}(b) \cap \mathbf{r}_B(x) \neq 0$ or $x\mathbf{r}(b) \cap aM \neq 0$. If $\mathbf{r}(b) \cap \mathbf{r}_B(x) \neq 0$, $\exists 0 \neq b' \in \mathbf{r}_B(b) \cap \mathbf{r}_B(x)$. Then, there is $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} \in \mathbf{r}_R(\alpha)$ because $\alpha \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & xb' \end{bmatrix} = 0$, which shows $\mathbf{r}_R(\alpha) \neq 0$. If $x\mathbf{r}(b) \cap aM \neq 0$, then $\exists 0 \neq ax' = xb' \in x\mathbf{r}(b) \cap aM$. Notice that $b' \neq 0$ and $x' \neq 0$. Then, there is $0 \neq \begin{bmatrix} 0 & -x' \\ 0 & b' \end{bmatrix} \in \mathbf{r}_R(\alpha)$ because $\alpha \begin{bmatrix} 0 & -x' \\ 0 & b' \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -x' \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & -ax' + xb' \\ 0 & bb' \end{bmatrix} = 0$, which shows $\mathbf{r}_R(\alpha) \neq 0$. Hence, " $\mathbf{r}_R(\alpha) \neq 0 \iff \mathbf{l}_R(\alpha) \neq 0$ " is vacuously true and we are done. \square

3.3 Upper Triangular Matrix Rings

This section investigates the relationship between the eversibility of a base ring R and the corresponding triangular matrix ring $\mathbb{T}_n(R)$.

First, we present some necessary conditions for $\mathbb{T}_n(R)$ to be eversible.

Proposition 3.3.1. *Let R be a ring and $n \geq 2$. If $\mathbb{T}_n(R)$ is left (resp. right) eversible, then $\mathbb{T}_{n-1}(R)$ is left (resp. right) eversible.*

Proof. If $n \geq 2$, we can write $\mathbb{T}_n(R) = \begin{bmatrix} \mathbb{T}_{n-1}(R) & R^{n-1} \\ 0 & R \end{bmatrix}$ (Here, R^{n-1} means an $(n-1) \times 1$ column vector). Say $A := \mathbb{T}_{n-1}(R)$, $M := R^{n-1}$, $B := R$. By Lemma 3.2.5, we know if $\mathbb{T}_n(R)$ is left eversible, then $A = \mathbb{T}_{n-1}(R)$ is also left eversible.

We can also write $\mathbb{T}_n(R) = \begin{bmatrix} R & R_{n-1} \\ 0 & \mathbb{T}_{n-1}(R) \end{bmatrix}$ (Here, R_{n-1} means an $1 \times (n-1)$ row vector). Say $A := R$, $M := R_{n-1}$, $B := \mathbb{T}_{n-1}(R)$. By Lemma 3.2.5 again, we know if $\mathbb{T}_n(R)$ is right eversible, then $B = \mathbb{T}_{n-1}(R)$ is also right eversible. \square

Corollary 3.3.2. *Let R be a ring and $n \geq 2$.*

- (1) *If $\mathbb{T}_n(R)$ is right eversible, then $xR \cap yR \neq 0$ for any $x, y \in zd_l^*(R)$.*
- (2) *If $\mathbb{T}_n(R)$ is left eversible, then $Rx \cap Ry \neq 0$ for any $x, y \in zd_r^*(R)$.*

(3) If $\mathbb{T}_n(R)$ is eversible, then $xR \cap yR \neq 0$ and $Rx \cap Ry \neq 0$ for any $x, y \in zd^*(R)$.

Proof. To prove (1): Suppose $\mathbb{T}_n(R)$ is right eversible. Since $n \geq 2$, we can apply Proposition 3.3.1 and know that $\mathbb{T}_2(R)$ is right eversible. Let $x, y \in zd_l^*(R)$ and consider $\alpha := \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \in \mathbb{T}_2(R)$. Note that $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = 0$, so $\mathbf{l}_{\mathbb{T}_2(R)}(\alpha) \neq 0$. Since $\mathbb{T}_2(R)$ is left eversible, we know $\mathbf{r}_{\mathbb{T}_2(R)}(\alpha) \neq 0$.

Assume for contradiction that $xR \cap yR = 0$. Let $0 \neq \beta = \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \in \mathbf{r}_{\mathbb{T}_2(R)}(\alpha)$.

Then $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} = \begin{bmatrix} xa' & xx' + yb' \\ 0 & 0 \end{bmatrix} = 0$, which gives us $xa' = 0$ and $xx' = y(-b')$. Note that $xx' = y(-b') \in xR \cap yR = 0$ and recall $x, y \in zd_l^*(R)$, we know $x' = 0$ and $b' = 0$. Also, since $xa' = 0$ and $x \in zd_l^*(R)$, we know $a' = 0$, and so $\beta = 0$, which contradicts $\beta \neq 0$. Thus, $xR \cap yR \neq 0$ for any $x, y \in zd_l^*(R)$.

To prove (2): The proof is similar to (1).

To prove (3): Suppose $\mathbb{T}_n(R)$ is eversible, then $xR \cap yR \neq 0$ for any $x, y \in zd_l^*(R)$ by (1), and $Rx \cap Ry \neq 0$ for any $x, y \in zd_r^*(R)$ by (2). Hence, $xR \cap yR \neq 0$ and $Rx \cap Ry \neq 0$ for any $x, y \in zd^*(R)$. □

Proposition 3.3.3. *Let R be a ring. If $\mathbb{T}_n(R)$ is eversible for some $n \geq 1$, then R is eversible.*

Proof. Suppose $\mathbb{T}_n(R)$ is eversible for some $n \geq 1$. Let $a \in Z_l(R)$, so there exists $0 \neq b \in R$ such that $ab = 0$. Say I_n is the identity matrix in $\mathbb{T}_n(R)$ and define matrices $A := aI_n \in \mathbb{T}_n(R)$, $B := bI_n \in \mathbb{T}_n(R)$. Then $AB = 0$ and so $A \in Z_l(\mathbb{T}_n(R))$. Recall $\mathbb{T}_n(R)$ is eversible, we know $A \in Z_r(\mathbb{T}_n(R))$, i.e. there exists $0 \neq C = [c_{ij}] \in \mathbb{T}_n(R)$ such that $[c_{ij}a] = CA = 0$. This means there exist i and j such that $c_{ij} \neq 0$ but $c_{ij}a = 0$, which means $a \in Z_r(R)$. Similarly, if $a \in Z_r(R)$, we can prove that $a \in Z_l(R)$. Therefore, R is eversible. □

Remark 3.3.4. However, the converse of 3.3.3 is not true. Here is an example that R is eversible but $\mathbb{T}_n(R)$ is not eversible for some $n \geq 2$:

Example 3.3.5. Let R be a domain that is not right Ore. Then R is eversible, but $\mathbb{T}_n(R)$ is not eversible for any $n \geq 2$.

Proof. We know that every domain is eversible. Fix $n \geq 2$. Assume for contradiction that $\mathbb{T}_n(R)$ is eversible, then by Corollary 3.3.2, we know $xR \cap yR \neq 0$ and $Rx \cap Ry \neq 0$ for any $x, y \in zd^*(R)$. Recall R is a domain, so $zd^*(R) = R \setminus \{0\}$, hence we know that for any $x, y \in R \setminus \{0\}$, there exist $r, r' \in R$ such that $xr = yr'$. (Here, r and r' cannot be 0 because x and y are both nonzero.) Hence, if $x \neq 0 \in R$ is arbitrary, then for any $y \in zd^*(R)$, there exist $r, r' \in R$ with $r \in R \setminus \{0\}$ such that $xr = yr'$. We still have an edge case: if $x = 0$, let $r' = 0 \in R$ and so for all $y \in zd^*(R)$, there exists $r \in zd^*(R)$ such that $0 = xr = yr'$. Therefore, for any $x \in R$ and $y \in zd^*(R)$, there exist $r \in zd^*(R)$ and $r' \in R$ such that $xr = yr'$. Hence, R is right Ore because R satisfies Definition 2.0.14, which contradicts that R is not right Ore. \square

Next, we characterize the eversibility of $\mathbb{T}_n(R)$ in the following two theorems:

Theorem 3.3.6. *Let R be a ring and $n \geq 2$. Then $\mathbb{T}_n(R)$ is eversible iff $\mathbb{T}_{n-1}(R)$ is eversible and, for any $A \in zd(\mathbb{T}_{n-1}(R))$, $a \in zd^*(R)$, $X \in \mathbb{M}_{n-1,1}(R)$ and $Y \in \mathbb{M}_{1,n-1}(R)$, the following hold:*

- (1) $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A) \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(X) \neq 0$ or $\mathbb{M}_{n-1,1}(R)a \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X \neq 0$.
- (2) $\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) \cap \mathbf{r}_{\mathbb{T}_{n-1}(R)}(Y) \neq 0$ or $a\mathbb{M}_{1,n-1}(R) \cap Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) \neq 0$.

Proof. (\Rightarrow) To prove the forward direction: Suppose $\mathbb{T}_n(R)$ is eversible. By Proposition 3.3.1, we know $\mathbb{T}_{n-1}(R)$ is eversible. Since $A \in zd(\mathbb{T}_{n-1}(R))$, we know there exist nonzero $A', A'' \in \mathbb{T}_{n-1}(R)$ such that $AA' = 0$ and $A''A = 0$.

Suppose for contradiction that (1) does not hold. That is, $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A) \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(X) = 0$ and $\mathbb{M}_{n-1,1}(R)a \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X = 0$. Then, for $\alpha := \begin{bmatrix} A & X \\ 0 & a \end{bmatrix} \in \mathbb{T}_n(R)$, there exists $0 \neq \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{T}_n(R)$ such that $\alpha \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & X \\ 0 & a \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} AA' & 0 \\ 0 & 0 \end{bmatrix} = 0$, which implies $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0$. However, for any $\begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \in \mathbf{l}_{\mathbb{T}_n(R)}(\alpha)$, we have $0 = \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \begin{bmatrix} A & X \\ 0 & a \end{bmatrix} = \begin{bmatrix} rA & rX + sa \\ 0 & ta \end{bmatrix}$. Since $rA = 0$, $r \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)$. Since $ta = 0$ and

$a \in zd^*(R)$, we know $t = 0$. Since $rX = -sa$ and $\mathbb{M}_{n-1,1}(R)a \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X = 0$, we know $rX = sa = 0$, then $sa = 0$ implies $s = 0$, and $rX = 0$ implies $r \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(X)$. Recall $r \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)$, so $r \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(X) \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A) = 0$. Thus, $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) = 0$, so $\mathbb{T}_n(R)$ is not eversible, which is a contradiction.

Suppose for contradiction that (2) does not hold. That is, $\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) \cap \mathbf{r}_{\mathbb{T}_{n-1}(R)}(Y) = 0$ and $a\mathbb{M}_{1,n-1}(R) \cap Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) = 0$. Then, for $\beta := \begin{bmatrix} a & Y \\ 0 & A \end{bmatrix} \in \mathbb{T}_n(R)$, there exists $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & A'' \end{bmatrix} \in \mathbb{T}_n(R)$ such that $\begin{bmatrix} 0 & 0 \\ 0 & A'' \end{bmatrix} \beta = \begin{bmatrix} 0 & 0 \\ 0 & A'' \end{bmatrix} \begin{bmatrix} a & Y \\ 0 & A \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A''A \end{bmatrix} = 0$, which implies $\mathbf{l}_{\mathbb{T}_n(R)}(\beta) \neq 0$. However, for any $\begin{bmatrix} u & v \\ 0 & w \end{bmatrix} \in \mathbf{r}_{\mathbb{T}_n(R)}(\beta)$, we have $0 = \begin{bmatrix} a & Y \\ 0 & A \end{bmatrix} \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} = \begin{bmatrix} au & av + Yw \\ 0 & Aw \end{bmatrix}$. Since $Aw = 0$, $w \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(A)$. Since $au = 0$ and $a \in zd^*(R)$, we know $u = 0$. Since $av = -Yw$ and $a\mathbb{M}_{1,n-1}(R) \cap Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) = 0$, we know $av = Yw = 0$, then $av = 0$ implies $v = 0$, and $Yw = 0$ implies $w \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(Y)$. Recall $w \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(A)$, so $w \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(Y) \cap \mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) = 0$. Thus, $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) = 0$, so $\mathbb{T}_n(R)$ is not eversible, which is a contradiction.

Therefore, (1) and (2) are valid.

(\Leftarrow) To prove the backward direction: Suppose $\mathbb{T}_{n-1}(R)$ is eversible and (1) & (2) both hold. We want to show $\mathbb{T}_n(R)$ is eversible. Let $\alpha \in \mathbb{T}_n(R)$, then we can write $\alpha = \begin{bmatrix} A & X \\ 0 & a \end{bmatrix}$, where $A \in \mathbb{T}_{n-1}(R)$, $X \in \mathbb{M}_{n-1,1}(R)$ and $a \in R$.

Case 1: If $A \in zd(\mathbb{T}_{n-1}(R))$. In this case, there exist nonzero $A', A'' \in \mathbb{T}_{n-1}(R)$ such that $AA' = 0$ and $A''A = 0$. Then we have $0 \neq \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{r}_{\mathbb{T}_n(R)}(\alpha)$ and $0 \neq \begin{bmatrix} A'' & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_{\mathbb{T}_n(R)}(\alpha)$. Thus, " $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ " holds because both sides have the same truth value.

Case 2: If $A \in zd^*(\mathbb{T}_{n-1}(R))$ and $a \in zd^*(R)$. For any $\begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \in \mathbf{r}_{\mathbb{T}_n(R)}(\alpha)$, $0 = \begin{bmatrix} A & X \\ 0 & a \end{bmatrix} \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} = \begin{bmatrix} Ar & As + Xt \\ 0 & at \end{bmatrix}$. Since $Ar = 0$, we have $r = 0$. Since $at = 0$, we have $t = 0$, and so $As = -Xt = 0$ implies $s = 0$. Hence, $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) = 0$. Similarly, we can show $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha)$. Thus, " $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) = 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) = 0$ " holds because both

sides have the same truth value.

Case 3: If $A \in zd^*(\mathbb{T}_{n-1}(R))$ and $a \in zd(R)$. Since $\mathbb{T}_{n-1}(R)$ is eversible for some $n \geq 2$, by Proposition 3.3.3, we know R is eversible. Since $a \in zd(R)$, there exist nonzero $a', a'' \in R$ such that $aa' = 0$ and $a''a = 0$. Then we have $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & a' \end{bmatrix} \in \mathbf{r}_{\mathbb{T}_n(R)}(\alpha)$ and $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & a'' \end{bmatrix} \in \mathbf{l}_{\mathbb{T}_n(R)}(\alpha)$. Thus, “ $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ ” holds because both sides have the same truth value.

Therefore, in all cases, $\mathbb{T}_n(R)$ is eversible.

□

Theorem 3.3.7. *Let R be a ring and $n \geq 2$. Then $\mathbb{T}_n(R)$ is eversible iff R is eversible and, for any $[a_{ij}] \in \mathbb{T}_n(R)$,*

- (1) *if $a_{11} \in zd(R)$ and $a_{ii} \in zd^*(R)$ for $i = 2, \dots, n$, then $X(a_{ij}) = 0$ has a non-trivial solution in R_n , and*
- (2) *if $a_{nn} \in zd(R)$ and $a_{ii} \in zd^*(R)$ for $i = 1, \dots, n - 1$, then $(a_{ij})X = 0$ has a non-trivial solution in R^n .*

Proof. (\Rightarrow) To prove the forward direction: Suppose $\mathbb{T}_n(R)$ is eversible. By Proposition 3.3.3, we know R is eversible. Let $\alpha = [a_{ij}] \in \mathbb{T}_n(R)$.

To prove (1), suppose $a_{11} \in zd(R)$ and $a_{ii} \in zd^*(R)$ for all $i = 2, \dots, n$. We know $\mathbf{r}_R(a_{11}) \neq 0$, then by Proposition 3.2.1 (1), $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0$. Since $\mathbb{T}_n(R)$ is eversible, we know $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$, so there exists $0 \neq [x_{ij}] \in \mathbb{T}_n(R)$ such that $0 = [x_{ij}][a_{ij}]$. By induction, we know $x_{ij} = 0$ for all $i \geq 2$ and $j \geq 1$. Hence, there must be some nonzero entry appearing in the first row of $[x_{ij}]$, and we know $X = [x_{11} \ \cdots \ x_{1n}]$ is a nontrivial solution of $X[a_{ij}] = 0$.

We can prove (2) in a similar way as (1).

(\Leftarrow) To prove the backward direction: Suppose R is eversible and (1) & (2) hold. If $[x_{11} \ \cdots \ x_{1n}]$ is a nontrivial solution of $X[a_{ij}] = 0$, then by (1), $x_{11} \neq 0$. If $[x_{11} \ \cdots \ x_{n1}]^T$ is a nontrivial solution of $[a_{ij}]X = 0$, then by (2), $x_{11} \neq 0$. Hence, for any $k = 1, \dots, n - 1$, every matrix in $\mathbb{T}_k(R)$ also satisfies (1) and (2).

To show $\mathbb{T}_n(R)$ is eversible, let $\alpha = [a_{ij}] \in \mathbb{T}_n(R)$ and we have 5 cases to discuss:

Case 1: If $a_{ii} \in zd^*(R)$ for all $i = 1, \dots, n$.

Claim 1: Under the condition of case 1, $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) = 0$.

Proof of claim 1: Let $\beta = [b_{ij}] \in \mathbf{r}_{\mathbb{T}_n(R)}(\alpha)$. Then $\alpha\beta = 0$. For any $i = 1, \dots, n$, since [the i^{th} -row of α][the i^{th} -col of β] = 0, we get $b_{ii} = 0$ for every $i = 1, \dots, n$. Then, consider [the 1^{th} -row of α][the 2^{th} -col of β] = 0, which gives $b_{12} = 0$, and consider [the 2^{th} -row of α][the 3^{th} -col of β] = 0, which gives $b_{23} = 0$. Repeat this step by considering [the i^{th} -row of α][the $(i+1)^{\text{th}}$ -col of β] = 0, we can get $b_{i,i+1} = 0$ for every $i = 1, \dots, n-1$. Next, consider [the i^{th} -row of α][the $(i+2)^{\text{th}}$ -col of β] = 0 from $i = 1$ to $i = n-1$, we can get $b_{i,i+2} = 0$ for every $i = 1, \dots, n-2$. Repeat this process and the final step is to consider [the i^{th} -row of α][the $(i+(n-1))^{\text{th}}$ -col of β] = 0 from $i = 1$ to $i = n-(n-1)$, we get $b_{i,i+(n-1)} = 0$ for every $i = 1, \dots, n-(n-1)$, that is, $b_{1,n} = 0$. Therefore, $\beta = 0$ and claim 1 is proved.

Claim 2: Under the condition of case 1, $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) = 0$.

Proof of claim 2: The proof is similar to claim 1.

By claims 1 and 2, we know “ $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) = 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) = 0$ ” holds under the case 1 because both sides have the same truth value.

Case 2: If $a_{ii} \in zd(R)$ for some $1 < i < n$. Note that we can write $\alpha = \begin{bmatrix} \alpha_1 & * \\ 0 & * \end{bmatrix}$,

where $\alpha_1 = \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ & \ddots & \vdots \\ & & a_{ii} \end{bmatrix} \in \mathbb{T}_i(R)$. Then $\mathbf{l}_{\mathbb{T}_i(R)}(\alpha_1) \neq 0$ by Proposition

3.2.2 (1). Since $\mathbb{T}_i(R)$ is eversible by induction, $\mathbf{r}_{\mathbb{T}_i(R)}(\alpha_1) \neq 0$, which follows that $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ by Proposition 3.2.1 (1). Note that we can also write $\alpha =$

$\begin{bmatrix} * & * \\ 0 & \alpha_2 \end{bmatrix}$, where $\alpha_2 = \begin{bmatrix} a_{ii} & \cdots & a_{in} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix} \in \mathbb{T}_{n-i+1}(R)$. Then $\mathbf{r}_{\mathbb{T}_{n-i+1}(R)}(\alpha_2) \neq 0$ by

Proposition 3.2.1 (1). Since $\mathbb{T}_{n-i+1}(R)$ is eversible by induction, $\mathbf{l}_{\mathbb{T}_{n-i+1}(R)}(\alpha_2) \neq 0$, which follows that $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ by Proposition 3.2.2 (1). Thus, “ $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ ” holds under the case 2.

Case 3: If $a_{11}, a_{nn} \in zd(R)$. Since $a_{11} \in zd(R)$, we know $\mathbf{r}_R(a_{11}) \neq 0$, then $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ by Proposition 3.2.1. Since $a_{nn} \in zd(R)$, we know $\mathbf{l}_R(a_{nn}) \neq 0$, then $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ by Proposition 3.2.2. Thus, “ $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ ” holds under the case 3.

0" holds under the case 3.

Case 4: If only $a_{11} \in zd(R)$. In this case, $\mathbf{r}_R(a_{11}) \neq 0$, then $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ by Proposition 3.2.1. Moreover, by our assumption (condition (1)), we know $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$. Thus, " $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ " holds under the case 4.

Case 5: If only $a_{nn} \in zd(R)$. In this case, $\mathbf{l}_R(a_{nn}) \neq 0$, then $\mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ by Proposition 3.2.2. Moreover, by our assumption (condition (2)), we know $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0$. Thus, " $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) \neq 0 \iff \mathbf{l}_{\mathbb{T}_n(R)}(\alpha) \neq 0$ " holds under the case 5.

In all cases, we showed $\mathbb{T}_n(R)$ is eversible.

□

Corollary 3.3.8. *Let R be a ring with $zd^*(R) = U(R)$ and $n \geq 1$. Then $\mathbb{T}_n(R)$ is eversible iff R is eversible.*

Proof. We only need to show that the eversibility of R implies the eversibility of $\mathbb{T}_n(R)$.

In the notation of Theorem 3.3.7, we want to check condition (1) holds here. Suppose $a_{11} \in zd(R)$ and $a_{ii} \in zd^*(R)$ for any $i = 2, \dots, n$. Let $X = [b_1 \ b_2 \ \cdots \ b_n] \in R_n$.

If we have $[b_1 \ b_2 \ \cdots \ b_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix} = 0$, then we get a system of equations

$$\begin{cases} b_1 a_{11} = 0 & (1) \\ b_1 a_{12} + b_2 a_{22} = 0 & (2) \\ b_1 a_{13} + b_2 a_{23} + b_3 a_{33} = 0 & (3) \\ \vdots & \vdots \\ b_1 a_{1n} + \cdots + b_n a_{nn} = 0 & (n) \end{cases}$$

To solve for X , we first choose $b_1 \neq 0$ such that $b_1 a_{11} = 0$. (Such b_1 exists because $a_{11} \in zd(R)$.) Since $a_{22} \in zd^*(R) = U(R)$, let $b_2 := -a_{22}^{-1} b_1 a_{12}$, which satisfies (2). Since $a_{33} \in zd^*(R) = U(R)$, let $b_3 := -a_{33}^{-1} (b_1 a_{13} + b_2 a_{23})$, which satisfies (3). Repeat this process. Since $b_1 \neq 0$, we know $X = [b_1 \ b_2 \ \cdots \ b_n]$ must be nontrivial. Hence, condition (1) in Theorem 3.3.7 holds here.

Similarly, we can show condition (2) in Theorem 3.3.7 also holds here.

Thus, by Theorem 3.3.7, we know $\mathbb{T}_n(R)$ is eversible.

□

Corollary 3.3.9. *Let R be a π -regular ring and $n \geq 1$. Then $\mathbb{T}_n(R)$ is eversible iff R is directly-finite.*

Proof. (\Rightarrow) To prove the forward direction: Suppose $\mathbb{T}_n(R)$ is eversible. By Proposition 3.3.3, R is eversible. Then by Proposition 2.13 in [6], we know R is directly-finite.

(\Leftarrow) To prove the backward direction: Suppose R is directly-finite. Since R is also π -regular, we know R is eversible by Proposition 3.1.2.

Claim: $zd^*(R) = U(R)$.

Proof of claim: Note that $U(R) \subseteq zd^*(R)$ is obvious. Now, let $a \in zd^*(R)$, so $\mathbf{r}_R(a) = 0$. Since R is π -regular, $a^m = a^m r a^m$ for some $r \in R$ and $m \geq 1$. Then $a^m(1 - r a^m) = 0$. Since $\mathbf{r}_R(a) = 0$, we have $1 - r a^m = 0$ by induction, hence $r a^m = 1$. That is, there exists $r a^{m-1} \in R$ such that $(r a^{m-1})a = 1$, which means $a \in U(R)$. We proved the claim.

By the above claim and Corollary 3.3.8, we conclude that $\mathbb{T}_n(R)$ is eversible.

□

Corollary 3.3.10. *Let R be a domain and $n \geq 2$. Then $\mathbb{T}_n(R)$ is eversible iff R is left and right Ore.*

Proof. (\Rightarrow) To show the forward direction: Suppose $\mathbb{T}_n(R)$ is eversible, then by Corollary 3.3.2, we know $xR \cap yR \neq 0$ and $Rx \cap Ry \neq 0$ for any $x, y \in zd^*(R)$. Since R is a domain, we know R is left and right Ore from the proof of Example 3.3.5.

(\Leftarrow) To show the backward direction: Suppose R is left and right Ore. Let $\alpha \in T := \mathbb{T}_n(R)$. By Proposition 2.0.15, we know R is a subring of its right fraction domain D . Note that D is a domain, then $S := \mathbb{T}_n(D)$ is a finite dimensional vector space over D , so S is artinian, hence S is eversible by [6, Example 2.5].

Suppose $\mathbf{r}_T(\alpha) = 0$, we want to show $\mathbf{l}_T(\alpha) = 0$. Let $\beta \in \mathbf{r}_S(\alpha)$, then there exists $0 \neq k \in R$ such that $\beta(kI_n) \in T$. (This operation is analogous to clearing denominators in all entries of a matrix in $\mathbb{T}_n(\mathbb{Q})$, turning it into a matrix in $\mathbb{T}_n(\mathbb{Z})$.)

Recall $\alpha\beta = 0$, then $\alpha\beta(kI_n) = 0$. Since $\mathbf{r}_T(\alpha) = 0$, we know $\beta(kI_n) = 0$, and so $\beta = 0$ as $k \neq 0$. Thus, $\mathbf{r}_S(\alpha) = 0$. Since S is eversible, we know $\mathbf{l}_S(\alpha) = 0$, hence $\mathbf{l}_T(\alpha) \subseteq \mathbf{l}_S(\alpha) = 0$.

Similarly, we can show $\mathbf{l}_T(\alpha) = 0$ implies $\mathbf{r}_T(\alpha) = 0$. Thus, $\mathbb{T}_n(R)$ is eversible. \square

Corollary 3.3.11. *Let R be a commutative ring. Then $\mathbb{T}_n(R)$ is eversible for all $n \geq 1$.*

Proof. We will show $\mathbb{T}_n(R)$ is eversible for any $n \geq 1$ by induction on n .

If $n = 1$, since R is commutative, R is eversible.

Assume $\mathbb{T}_{n-1}(R)$ is eversible for some $n \geq 2$. Let $A \in \text{zd}(\mathbb{T}_{n-1}(R))$, $a \in \text{zd}^*(R)$, $X \in \mathbb{M}_{n-1,1}(R)$ and $Y \in \mathbb{M}_{1,n-1}(R)$.

Claim 1: Condition (1) in Theorem 3.3.6 holds.

Proof of claim 1: We have two cases to discuss.

- **Case I:** If $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X = 0$. Since $A \in \text{zd}(\mathbb{T}_{n-1}(R))$, there exists $0 \neq A' \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)$. Note that $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X = 0$ implies $A'X = 0$, which means $A' \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(X)$. Hence, $0 \neq A' \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A) \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(X)$, and so Theorem 3.3.6 (1) holds when $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X = 0$.
- **Case II:** If $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X \neq 0$. Say $A'X \neq 0$, where $A' \in \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)$. Since $a \in \text{zd}^*(R)$, we know $0 \neq (A'X)a \in \mathbb{M}_{n-1,1}(R)a$. Since R is commutative, $(A'X)a = a(A'X)$, where aA' is still in $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)$. Thus, $0 \neq (A'X)a \in \mathbb{M}_{n-1,1}(R)a \cap \mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X$, and so Theorem 3.3.6 (1) holds when $\mathbf{l}_{\mathbb{T}_{n-1}(R)}(A)X \neq 0$.

Claim 2: Condition (2) in Theorem 3.3.6 holds.

Proof of claim 2: We have two cases to discuss.

- **Case I:** If $Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) = 0$. Since $A \in \text{zd}(\mathbb{T}_{n-1}(R))$, there exists $0 \neq A' \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(A)$. Note that $Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) = 0$ implies $YA' = 0$, which means $A' \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(Y)$. Hence, $0 \neq A' \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) \cap \mathbf{r}_{\mathbb{T}_{n-1}(R)}(Y)$, and so Theorem 3.3.6 (2) holds when $Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) = 0$.

- **Case II:** If $Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) \neq 0$. Say $YA' \neq 0$, where $A' \in \mathbf{r}_{\mathbb{T}_{n-1}(R)}(A)$. Since $a \in zd^*(R)$, we know $0 \neq a(YA') \in a\mathbb{M}_{1,n-1}(R)$. Since R is commutative, $a(YA')a = (YA')a$, where $A'a$ is still in $\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A)$. Thus, $0 \neq a(YA') \in a\mathbb{M}_{1,n-1}(R) \cap Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A)$, and so Theorem 3.3.6 (2) holds when $Y\mathbf{r}_{\mathbb{T}_{n-1}(R)}(A) \neq 0$.

By claim 1, claim 2 and the inductive hypothesis, we can apply Theorem 3.3.6 to conclude that $\mathbb{T}_n(R)$ is eversible.

□

3.4 Power Series and Polynomial Rings

In this section, we explore how the eversibility of $R[t]$ and $R[[t]]$ relates to that of R , including several partial results and examples illustrating non-equivalence.

First, we investigate eversibility in formal power series rings.

Proposition 3.4.1. *If $R[[x]]$ is eversible, then R is eversible.*

Proof. Let $a \in R$ be a left zero-divisor. Then a is also a left zero-divisor in $R[[x]]$. Since $R[[x]]$ is eversible, a is also a right zero-divisor in $R[[x]]$, hence, there exists $0 \neq f(x) \in R[[x]]$ such that $f(x)a = 0$. Say $f = b_0 + b_1x + b_2x^2 + \dots$. Then $b_0a + b_1xa + b_2x^2a + \dots = 0$, which implies $b_i a = 0$ for every $i \geq 0$. Since $f(x) \neq 0$, we know there must be some $j \geq 0$ such that $b_j \neq 0$. Hence, a is a right zero-divisor of R . Similarly, we can show that a right zero-divisor in R must be a left zero-divisor in R . Thus, R is eversible.

□

Remark 3.4.2. However, the converse statement is false and we have a counterexample here:

Example 3.4.3. There exists a domain R that is both left Ore and right Ore in [11, Example 10.31A]. Hence, $S := \mathbb{T}_n(R)$ is eversible by Corollary 3.3.10. However, $R[[x]]$ is not right Ore by [11, Example 10.31A], then by Corollary 3.3.2, we know $\mathbb{T}_n(R[[x]])$ is not eversible, hence, $S[[x]] = \mathbb{T}_n(R)[[x]]$ is not eversible.

We now turn our attention to the eversibility of polynomial rings.

Proposition 3.4.4. *If $R[x]$ is eversible, then R is eversible.*

Proof. Let $a \in R$ be a left zero-divisor. Then a is also a left zero-divisor in $R[x]$. Since $R[x]$ is eversible, a is also a right zero-divisor in $R[x]$, hence, there exists $0 \neq f(x) \in R[x]$ such that $f(x)a = 0$. Say $f = b_0 + b_1x + \cdots + b_mx^m$. Then $b_0a + b_1xa + \cdots + b_mx^ma = 0$, which implies $b_ia = 0$ for every $i = 0, \dots, m$. Since $f(x) \neq 0$, we know there must be some j such that $b_j \neq 0$. Hence, a is a right zero-divisor of R .

Similarly, we can show that a right zero-divisor in R must be a left zero-divisor in R . Thus, R is eversible. □

Remark 3.4.5. However, it remains an open question whether the eversibility of R implies that of $R[x]$. We conjecture that this is not the case.

The following example involves a noncommutative, eversible base ring R for which $R[t]$ is also eversible. Hence, it does not yield a counterexample to the conjectured failure of eversibility under polynomial extension. From this example, we know that additional conditions on R are required to break the eversibility of $R[t]$.

Example 3.4.6. Consider $R = \mathbb{T}_2(\mathbb{Z}_2)$, which is noncommutative. However, $R[t]$ is eversible. Indeed, $R[t] = \mathbb{T}_2(\mathbb{Z}_2)[t] \cong \mathbb{T}_2(\mathbb{Z}_2[t])$, which is eversible by Corollary 3.3.11 because $\mathbb{Z}_2[t]$ is commutative.

3.5 Counterexamples to Existing Results

We conclude with counterexamples to prior claims in [6], especially regarding formal triangular matrix rings and their proposed criteria for eversibility.

First, let us look at [6, Theorem 3.3]: Let A, B be rings and $M = {}_A M_B$ be a bimodule. Suppose $m \neq 0$ and $D = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$ is an element of the formal triangular matrix $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$. Then the following statements are equivalent:

- (1) D is a left zero-divisor.
- (2) At least one of the following occurs:
 - (a) $a \in Z_l(A)$.
 - (b) $a \in Z_l^*(A)$, $b \in Z_l(B)$ and $\exists 0 \neq b'' \in \mathbf{r}(b)$ such that $mb'' = 0$.
 - (c) $a \in Z_l^*(A)$, $b \in Z_l^*(B)$ and $\exists 0 \neq m'' \in M$ such that $am'' = 0$.

In fact, there is a logical gap in the proof of the above result, specifically in part (b), which calls the validity of the entire statement into question. Here, we provide a counterexample to this statement.

Example 3.5.1. Consider \mathbb{Z}_3 as a \mathbb{Z}_6 -bimodule via $[r]_6 \cdot [m]_3 := [rm]_3$ and $[m]_3 \cdot [r]_6 := [mr]_3$. Let $A = B = \mathbb{Z}_6$ and $M = \mathbb{Z}_3$. Let $D = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$, then there exists $0 \neq \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \in R$ such that $D \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = 0$, so D is a left zero-divisor in R . Note that this is in case (b) because $a = [5]_6 \in Z_l^*(A)$, $b = [3]_6 \in Z_l(B)$.

Suppose for contradiction that [6, Theorem 3.3 (b)] was correct, then $\exists 0 \neq b'' \in \mathbf{r}([3]_6) = \{[0]_6, [2]_6, [4]_6\}$ such that $mb'' = [1]_3 b'' = [0]_3$. Note that $[1]_3 b'' = [0]_3$ implies $b'' = [3]_6$ or $b'' = [0]_6$. Note that $0 \neq b'' \in \mathbf{r}([3]_6)$ implies $b'' \neq [3]_6$ and $b'' \neq [0]_6$, which gives a contradiction. Hence, [6, Theorem 3.3] is not correct.

Moreover, we can give another counterexample showing that [6, Theorem 3.3 and 3.4] are incorrect:

Example 3.5.2. Let $R = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ where A is a ring with a central non-trivial idempotent e . Let $\alpha = \begin{bmatrix} 1 & 1-e \\ 0 & e \end{bmatrix} \in R$ and $\beta = \begin{bmatrix} e & 1-e \\ 0 & 1 \end{bmatrix} \in R$. Then

- (1) $\mathbf{r}_R(\alpha) \neq 0$ but condition (2) of [6, Theorem 3.3] does not hold.
- (2) $\mathbf{r}_R(\beta) \neq 0$ but condition (2) of [6, Theorem 3.4] does not hold.

Proof. To prove (1): In the notation of [6, Theorem 3.3], $a = 1$, $m = 1 - e$ and $b = e$. Note that $a = 1 \in zd^*(A)$ and there exists $0 \neq 1 - e \in A$ such that $(1 - e)b = (1 - e)e = e - e = 0$, so $b \in zd(A)$. Hence, this is in the case (b) of [6, Theorem 3.3]. Now, suppose

for contradiction that [6, Theorem 3.3] holds. Then there exists $0 \neq b'' \in A$ such that $bb'' = 0$ and $mb'' = 0$. Hence, $0 = mb'' = (1-e)b'' = b'' - eb'' = b'' - bb'' = b'' - 0 = b''$, which contradicts to $b'' \neq 0$.

To prove (2): In the notation of [6, Theorem 3.4], $a = e$, $m = 1-e$, $b = 1$. Note that $b = 1 \in zd^*(A)$ and there exists $0 \neq 1-e \in A$ such that $a(1-e) = e(1-e) = e-e = 0$, so $a \in zd(A)$. Hence, this is in the case (b) of [6, Theorem 3.4]. Now, suppose for contradiction that [6, Theorem 3.4] holds. Then there exists $0 \neq a'' \in A$ such that $a''a = 0$ and $a''m = 0$. Hence, $0 = a''m = a''(1-e) = a'' - a''e = a'' - a''a = a'' - 0 = a''$, which contradicts to $a'' \neq 0$.

□

Remark 3.5.3. We just showed that [6, Theorem 3.3 and 3.4] are incorrect. The fixed version of [6, Theorem 3.3] is our Proposition 3.2.1. Also, the fixed version of [6, Theorem 3.4] is our Proposition 3.2.2.

Next, we want to disprove [6, Proposition 3.5].

In order to construct counterexamples to [6, Proposition 3.5], we need the following lemma:

Lemma 3.5.4. Let $X = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$ and $Y = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Let $x \in X$ and $y \in Y$. Then

$$(1) \mathbf{r}_X(x) = 0 \iff x = \begin{bmatrix} \bar{1} & * \\ 0 & n \end{bmatrix} \text{ for some } n \neq 0.$$

$$(2) \mathbf{l}_Y(y) = 0 \iff y = \begin{bmatrix} n & * \\ 0 & \bar{1} \end{bmatrix} \text{ for some } n \neq 0.$$

(3) X is left eversible but not right eversible.

(4) Y is right eversible but not left eversible.

Proof. To prove (1): Let $x = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $y = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$. Note that “ $xy = 0$ ” is equivalent to “ $aa' = 0$ and $ab' + bc' = 0$ and $cc' = 0$ ”. If $\mathbf{r}_X(x) = 0$, then $a \neq 0$ because if $a = 0$, then $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{r}_X(x)$, contradicting $\mathbf{r}_X(x) = 0$. However, since $a \in \mathbb{Z}_2$, $a \neq 0 \implies a = 1$. Similarly, if $\mathbf{r}_X(x) = 0$, then $c \neq 0$. Thus, the forward

direction of (1) holds. Conversely, suppose $x = \begin{bmatrix} \bar{1} & * \\ 0 & n \end{bmatrix}$ for some $0 \neq n \in \mathbb{Z}$. Let $y = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \in \mathbf{r}_X(x)$, so $0 = xy = \begin{bmatrix} \bar{1} & * \\ 0 & n \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$ gives us “ $a' = 0$ and $b' + *c' = 0$ and $nc' = 0$ ”. Since $n \neq 0$ and $nc' = 0$, we have $c' = 0$. Since $b' + *c' = 0$, we have $b' = 0$. Thus, $a' = b' = c' = 0$, which means $y = 0$, and so $\mathbf{r}_X(x) = 0$.

The proof of (2) is similar to (1).

To prove (3): To show X is not right eversible, consider $x_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Note that $\mathbf{l}_X(x_1) \neq 0$ since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_X(x_1)$. If $y = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \in \mathbf{r}_X(x_1)$, then $0 = x_1y$ gives us $a' = 0$, $b' = 0$ and $2c' = 0$, which means $y = 0$ and so $\mathbf{r}_X(x_1) = 0$. Thus, $\mathbf{l}_X(x_1) \neq 0$ but $\mathbf{r}_X(x_1) = 0$, i.e. X is not right eversible. To show X is left eversible, suppose $x = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbf{zd}_l(X)$, i.e. $\mathbf{r}_X(x) \neq 0$. We need to show $\mathbf{l}_X(x) \neq 0$. If c is even, then $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = 0$, which means $0 \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_X(x)$. If c is odd, since $\mathbf{r}_X(x) \neq 0$, we know $a = 0$ by (1). Thus, $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b+bc \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b+bc \\ 0 & 0 \end{bmatrix}$. Since c is odd, $c \equiv 1 \pmod{2}$, so $b+bc \equiv b+b \equiv 0 \pmod{2}$. Thus, $0 \neq \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_X(x)$, and so X is left eversible.

(4) is exactly shown in [6, Example 2.2].

□

Now, we look back at [6, Proposition 3.5]: “Let A, B be rings and $M = {}_A M_B$ be a bimodule. If the formal triangular matrix $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is eversible, then A and B are eversible rings.” Based on the above Lemma 3.5.4, we can construct a counterexample:

Example 3.5.5. Let $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix} & \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \\ 0 & \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \end{bmatrix}$. Then T is eversible,

but B is not left eversible.

Proof. First, by above Lemma 3.5.4, we know B is not left eversible.

Now, we want to show T is eversible. Let $\alpha = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in T$. We know that A is a finite ring, so A is eversible by Proposition 3.1.1.

Note that every element in $A = T_2(\mathbb{Z}_2)$ is either a unit or a zero divisor, because if $a \in A$ is not a unit, then the rank of a is 1 or 0, and so there must be at least one 0 appearing at the diagonal of a , i.e., a is of the form $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & * \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$, which are indeed zero divisors in A .

Claim 1: If $a \in zd^*(A)$ and $\mathbf{l}_B(b) \neq 0$, then $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$.

Proof of Claim 1: Since $\mathbf{l}_B(b) \neq 0$, we know $\mathbf{l}_T(\alpha) \neq 0$ by (1) in Proposition 3.2.2. Then we want to prove $\mathbf{r}_T(\alpha) \neq 0$. Since $\mathbf{l}_B(b) \neq 0$, b is a right zero-divisor in B . By Lemma 3.5.4 (4), we know B is right eversible, hence b is a left zero-divisor in B , which means $\mathbf{r}_B(b) \neq 0$. Then we can choose $0 \neq b' \in \mathbf{r}_B(b)$ and consider $\beta := \begin{bmatrix} 0 & -a^{-1}xb \\ 0 & b' \end{bmatrix} \in T$. (Here, a^{-1} exists as $a \in zd^*(A)$.) Note that $\alpha\beta = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -a^{-1}xb \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & bb' \end{bmatrix} = 0$ and $\beta \neq 0$, hence $\mathbf{r}_T(\alpha) \neq 0$. Thus, $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$ holds because both sides have the same truth value. Hence, claim 1 is proved.

Claim 2: If $a \in zd^*(A)$ and $\mathbf{l}_B(b) = 0$, then $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$.

Proof of Claim 2: By 3.5.4, $\mathbf{l}_B(b) = 0$ implies $b = \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix}$ for some $n \neq 0$ and $\bar{k} \in \mathbb{Z}_2$. We will show $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$ holds whenever n is even or odd:

(i) If n is even, note that $b' := \begin{bmatrix} 0 & \bar{1} \\ 0 & \bar{0} \end{bmatrix} \in \mathbf{r}_B(b)$ (because $bb' = \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix} \begin{bmatrix} 0 & \bar{1} \\ 0 & \bar{0} \end{bmatrix} = \begin{bmatrix} 0 & \bar{n} \\ 0 & \bar{0} \end{bmatrix} = 0$), then we have $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -a^{-1}xb' \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & -xb' + xb' \\ 0 & bb' \end{bmatrix} = 0$, which implies $\mathbf{r}_T(\alpha) \neq 0$. Next, we will show $\mathbf{l}_T(\alpha) \neq 0$: note that $x' := \begin{bmatrix} \bar{1} & \bar{k} \\ 0 & \bar{0} \end{bmatrix} \in \mathbf{l}_M(b)$

(because $x'b = \begin{bmatrix} \bar{1} & \bar{k} \\ 0 & \bar{0} \end{bmatrix} \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{n} & \overline{2k} \\ 0 & 0 \end{bmatrix} = 0$), then we have $\begin{bmatrix} 0 & x' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & x'b \\ 0 & 0 \end{bmatrix} = 0$, which implies $\mathbf{l}_T(\alpha) \neq 0$. Thus, if n is even, $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$ holds because both sides have the same truth value.

(ii) If n is odd, for any $\begin{bmatrix} n' & \bar{s} \\ 0 & \bar{t} \end{bmatrix} \in \mathbf{r}_B(b)$, we know $0 = \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix} \begin{bmatrix} n' & \bar{s} \\ 0 & \bar{t} \end{bmatrix} = \begin{bmatrix} nn' & \overline{ns + kt} \\ 0 & \bar{t} \end{bmatrix}$, then $n' = 0$, $\bar{t} = \bar{0}$ and $\bar{s} = \bar{0}$ (because $ns \in \mathbb{Z}$ must be even), which means $\mathbf{r}_B(b) = 0$. Then, for any $\begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} \in \mathbf{r}_T(\alpha)$, we know $0 = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix} = \begin{bmatrix} aa' & ax' + xb' \\ 0 & bb' \end{bmatrix}$, and so $a' = 0$, $b' = 0$ and $x' \in \mathbf{r}_{T_2(\mathbb{Z}_2)}(a)$. However, we recall $a \in zd^*(T_2(\mathbb{Z}_2))$, so $x' = 0$, hence $\mathbf{r}_T(\alpha) = 0$. Next, we will show $\mathbf{l}_T(\alpha) = 0$: for any $\begin{bmatrix} \bar{r} & \bar{s} \\ 0 & \bar{t} \end{bmatrix} \in \mathbf{l}_M(b)$, we have $0 = \begin{bmatrix} \bar{r} & \bar{s} \\ 0 & \bar{t} \end{bmatrix} \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{r}n & \overline{rk + s} \\ 0 & \bar{t} \end{bmatrix}$, so $\bar{r} = \bar{0}$, $\bar{s} = \bar{0}$ and $\bar{t} = \bar{0}$, which means $\mathbf{l}_M(b) = 0$. Recall our assumption that $\mathbf{l}_B(b) = 0$, and $a \in zd^*(A)$ gives us $\mathbf{l}_A(a) = 0$, therefore, by Proposition 3.2.2, we know $\mathbf{l}_T(\alpha) = 0$. Thus, if n is odd, $\mathbf{l}_T(\alpha) = 0 \iff \mathbf{r}_T(\alpha) = 0$ holds because both sides have the same truth value.

Hence, claim 2 is proved.

Claim 3: If $a \in zd(A)$, then $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$.

Proof of Claim 3: Since $a \in zd(A) = zd(T_2(\mathbb{Z}_2))$ and the set of all nonzero zero-divisor in A is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$, then we know $\mathbf{r}_A(a) \neq 0$ by computing all possible values of a . Then, by Proposition 3.2.1 (1), we know $\mathbf{r}_T(\alpha) \neq 0$.

Next, we will show $\mathbf{l}_T(\alpha) \neq 0$. If $\mathbf{l}_B(b) \neq 0$, by Proposition 3.2.2 (1), we know $\mathbf{l}_T(\alpha) \neq 0$ and we are done. Therefore, we may assume $\mathbf{l}_B(b) = 0$ from now on. By Lemma 3.5.4 (2), $\mathbf{l}_B(b) = 0$ implies $b = \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix}$ for some $n \neq 0$ and $\bar{k} \in \mathbb{Z}_2$. We will show $\mathbf{l}_T(\alpha) \neq 0$ whenever n is even or odd:

(i) If n is even: note that $x' := \begin{bmatrix} \bar{1} & \bar{k} \\ 0 & \bar{0} \end{bmatrix} \in \mathbf{l}_M(b)$ (because $\begin{bmatrix} \bar{1} & \bar{k} \\ 0 & \bar{0} \end{bmatrix} \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{n} & \overline{2k} \\ 0 & \bar{0} \end{bmatrix} = 0$), then the element $0 \neq \begin{bmatrix} 0 & x' \\ 0 & 0 \end{bmatrix} \in \mathbf{l}_T(\alpha)$ (because $\begin{bmatrix} 0 & x' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} =$

$\begin{bmatrix} 0 & x'b \\ 0 & 0 \end{bmatrix} = 0$). Hence, $\mathbf{l}_T(\alpha) \neq 0$ when n is even.

(ii) If n is odd: consider $u := \begin{bmatrix} \bar{1} & \bar{k} \\ 0 & \bar{1} \end{bmatrix} \in A$. Clearly, $u \in U(A)$, hence u^{-1} exists.

We observe that for any $y = \begin{bmatrix} \bar{r} & \bar{s} \\ 0 & \bar{t} \end{bmatrix} \in M$, $yb = yu$ because n is odd and

$$yb = \begin{bmatrix} \bar{r} & \bar{s} \\ 0 & \bar{t} \end{bmatrix} \begin{bmatrix} n & \bar{k} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{r}n & \overline{rk+s} \\ 0 & \bar{t} \end{bmatrix}, \quad yu = \begin{bmatrix} \bar{r} & \bar{s} \\ 0 & \bar{t} \end{bmatrix} \begin{bmatrix} \bar{1} & \bar{k} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{r} & \overline{rk+s} \\ 0 & \bar{t} \end{bmatrix}.$$

Recall $a \in zd(A)$, we know $\mathbf{l}_A(a) \neq 0$, then we can choose $0 \neq a' \in \mathbf{l}_A(a)$. Then the element $\beta := \begin{bmatrix} a' & a'xu^{-1} \\ 0 & 0 \end{bmatrix}$ is nonzero. Note that $\beta\alpha = \begin{bmatrix} a' & a'xu^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} a'a & a'x + (a'xu^{-1})b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a'a & a'x + (a'xu^{-1})u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar{2}a'x \\ 0 & 0 \end{bmatrix} = 0$, hence, $\mathbf{l}_T(\alpha) \neq 0$ when n is odd.

Thus, if $a \in zd(A)$, $\mathbf{l}_T(\alpha) \neq 0 \iff \mathbf{r}_T(\alpha) \neq 0$ holds because both sides have the same truth value. Hence, claim 3 is proved.

Finally, by claims 1, 2 and 3, we conclude that T is eversible. □

Example 3.5.6. Let $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix} & \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \\ 0 & \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \end{bmatrix}$. Then T is eversible,

but A is not right eversible.

Proof. The proof is similar to the above Example 3.5.5. □

Remark 3.5.7. We just constructed two counterexamples of [6, Proposition 3.5]. The fixed version of [6, Proposition 3.5] is our Lemma 3.2.5.

Next, we will show that [6, Theorem 3.7] is actually incorrect. (In particular, the converse direction is wrong). Indeed, “there exists $1 \leq k \leq n$ with $a_{kk} \in zd_l(R)$ ”

cannot imply “ $A = [a_{ij}]$ is a left zero-divisor”. Also, “there exists $1 \leq k \leq n$ with $a_{kk} \in zd_r(R)$ ” cannot imply “ $A = [a_{ij}]$ is a right zero-divisor”. Regarding these two incorrect statements, we construct a counterexample of them:

Example 3.5.8. Let R be a domain that is not right Ore (i.e. for some nonzero $x, y \in R$, $xR \cap yR = 0$). Let $\alpha :=$

$$\alpha := \begin{bmatrix} x & y & \cdots & y \\ & x & y & \vdots \\ & & \ddots & \ddots \\ & & & x & y \\ & & & & 0 \end{bmatrix} \text{ and } \beta := \begin{bmatrix} 0 & y & \cdots & y \\ & x & y & \vdots \\ & & \ddots & \ddots \\ & & & x & y \\ & & & & x \end{bmatrix} \in \mathbb{T}_n(R).$$

Then $\mathbf{r}_{\mathbb{T}_n(R)}(\alpha) = 0$ and $\mathbf{l}_{\mathbb{T}_n(R)}(\beta) = 0$. Hence, α is not a left zero-divisor but its diagonal contains an entry $0 \in zd_l(R)$, and β is not a right zero-divisor but its diagonal contains an entry $0 \in zd_l(R)$.

The following provides another counterexample to [6, Proposition 3.7 (1)]:

Example 3.5.9. Let $R = k\langle x, y \rangle / (x^2y)$, where k is an arbitrary base field. Consider

$A := \begin{bmatrix} y^2 + x + 1 & 1 \\ 0 & x \end{bmatrix} \in \mathbb{T}_2(R)$, then A is not a left zero-divisor of $\mathbb{T}_2(R)$. However, x is a left zero-divisor appearing on the diagonal of A , which shows [6, Proposition 3.7 (1)] is incorrect.

Proof. Since [6, Proposition 3.7] does not impose any conditions on the ring R , the choice $R = k\langle x, y \rangle / (x^2y)$ is a valid context for constructing a counterexample. Hence, we only need to show that A is not a left zero-divisor of $\mathbb{T}_2(R)$.

Suppose for contradiction that there exists $0 \neq B = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathbb{T}_2(R)$ such that

$$AB = 0, \text{ then } 0 = \begin{bmatrix} y^2 + x + 1 & 1 \\ 0 & x \end{bmatrix} \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} = \begin{bmatrix} (y^2 + x + 1)a & (y^2 + x + 1)m + b \\ 0 & xb \end{bmatrix}.$$

Since $(y^2 + x + 1)a = 0$, we have $a = 0$ because $\mathbf{r}_R(y^2) = 0$.

Since $xb = 0$, we discuss two cases:

If $b = 0$, then $(y^2 + x + 1)m = 0$, so $m = 0$ because $\mathbf{r}_R(y^2) = 0$. Hence, $B = 0$, which contradicts to $B \neq 0$.

If $b \neq 0$, then $b = xyr$ for some $0 \neq r \in R$ because $\mathbf{r}_R(x) = xyR$. Hence, $(y^2 + x + 1)m = xyr$. However, there is no solution for m (because the term y^2m

cannot be canceled or be expressed as an element in xyR). Thus, $b \neq 0$ is impossible.

Therefore, $\mathbf{r}_{\mathbb{T}_2R}(A) = 0$, which means A is not a left zero-divisor.

□

Finally, the forward direction of [6, Proposition 3.9 (1)] is incorrect. In fact, we discussed that the eversibility of R cannot imply the eversibility of $T_n(R)$. We have constructed a counterexample (see Example 3.3.5) of [6, Proposition 3.9 (1)]. The fixed version is our Theorem 3.3.7.

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