Estimation of a Cluster-Based Correlation Model for Familial-Spatial and Spatial-Temporal Continuous Data

by

© **Sahar Arshadi**

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Department of Mathematics and Statistics

Memorial University

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Abstract

The study of data collected from geographical regions is called spatial data analysis, and the study of these data over time is called spatial-temporal data analysis. Recently, the analysis of spatial and spatial-temporal data has been of interest to researchers in the fields of epidemiology, biology, forestry, agriculture, and geography. In the analysis of spatial data, locations within a user-specified distance are usually considered to constitute a cluster. Responses obtained from neighbouring locations are likely to be correlated due to the effect of latent variables at each location. Many authors have used a linear mixed effects regression model to analyze continuous/Gaussian spatial data under a variety of assumptions between and within clusters. Previous studies have focused on the single observation studies obtained from each location (Mariathas and Sutradhar (2016)). The intent of this research is to consider an extension to multivariate data collected at each location that we refer to as familial-spatial data. Thus, aside from the correlation between responses from neighbouring locations, we also consider the effect of the familial correlation between the multivariate responses collected at the same location. We then develop a familial-spatial cluster-based correlation model for the data and propose methods for estimating the model parameters. Furthermore, we extend the cluster-based correlation idea of Mariathas and Sutradhar (2016) to develop models to spatial-temporal data. Intensive simulation studies are applied to assess the performance of the models.

Dedication

To my late father, Hossein Arshadi, who was the origin of my success.

To my mother, Marjan Tabibzadeh-Ghamsari, for her ongoing love and support.

To my husband, Babak Mofid, who is the love of my life and for all his support.

To my brothers, Ali and Amir, who have always been by my side all these years.

Lay summary

In many spatial studies, the aim is to identify the correlation between the responses from the same or different locations, families, and time points. For example, individuals living in the same or different households may experience lead poisoning in their blood. Blood lead level can be affected by not only some fixed covariates such as age, gender, water hardness yet unobserved variables from neighboring locations and a shared family latent variable based on the area and family to which people belong. Or in an environmental study, the amount of specific particulate in the water every week may be influenced by some latent variables from adjacent locations. Therefore, to determine the correlation between the two response variables, it is important to know how far apart these two responses are in terms of their spatial distance. In this study, any nearby locations within a pre-specified distance belong to a cluster. In the first model, we consider a familial-spatial linear model when multivariate responses at the same location belong to a family. The second model considers a spatial-temporal model when there is an autoregressive order of one structure between the responses. We proposed different statistical estimation methods to evaluate the performance of the parameters of the two models.

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Statement of contribution

Dr. Alwell Oyet proposed the research question that was investigated throughout this thesis. The overall study was jointly designed by Dr. Alwell Oyet, Dr. Asokan Mulayath Variyath and Sahar Arshadi. The algorithms were implemented, the simulation studies were conducted and the manuscript was drafted by Sahar Arshadi. Dr. Alwell Oyet and Dr. Asokan Mulayath Variyath jointly supervised the study and contributed to the final manuscript.

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List of abbreviations

AR(1)	Autoregressive Order One
ARMA	Autoregressive Moving Average
BLL	Blood Lead Level
FRF	Fixed Rank Filtering
GLS	Generalized Least Square
GQL	Generalized Quasi-Likelihood
ML	Maximum Likelihood
MM	Method of Moment
OLS	Ordinary Least Square
REML	Restricted Maximum Likelihood
SM	Simulated Mean
SSE	Simulated Standard Error
SRE	Spatial Random Effect
STRE	Spatial-Temporal Random Effect
VAR	Vector Autoregressive

Chapter 1

An Overview of Statistical Analysis of Spatial and Spatial-Temporal Data

In many epidemiological and environmental studies, the correlation between responses from different locations or different locations and moments in time is of interest. Spatial data refers to data collected from different locations and spatial-temporal data refers to those data collected from different locations at different moments in time. Recently, Mariathas and Sutradhar (2016) proposed a pair-wise linear (continuous) mixed effect model and compared the method of moment (MM) and maximum likelihood (ML) estimates of regression effects when there is correlation between any two responses from different locations. These random measurements obtained from different locations are referred to as *spatial data*, and the correlations between them are known as *spatial correlations*. For example, a researcher may be interested in evaluating the effects of individual, epidemiological, and environmental covariates such as age, gender, and water hardness on the blood lead level (BLL) of residents. Variations in the BLL and correlations among the BLLs of individuals captured from different areas (locations) are important parameters to prevent the risk of high BLL in the next generation. Along with these fixed covariates, latent variables from nearby regions that cannot be measured directly may affect these responses. Such latent variables from different locations that affect the response will be referred to as *location random effects*.

On a continuous scale with a single observation at each location, Mariathas and Sutradhar (2016), with respect to the linear distance between the locations, considered the impact of neighbouring location random effects on the response and constructed the pair-wise correlations between the location random effects, and following on that built the spatial correlations between the responses from these locations. They estimated the impact of regression covariates after modelling the spatial correlation between the responses.

In the first part of this thesis, we develop an extension to Mariathas and Sutradhar (2016) model for analyzing multivariate observations at each location. Aside from the spatial correlation between nearby locations, we also assume that there exist familial correlation between the multivariate observations at a given location induced by a common unobservable random effect. Mariathas and Sutradhar (2016) used the terms *cluster* and *family* interchangeably, but throughout this work, it is essential to emphasize that a cluster is not interchangeable with a family. From now on, cluster will be used to refer to a group of nearby locations within a user-specified distance d, and family to a group of observations at each location. The family of observations collected at each location and the correlations between the responses from the same or different locations are referred to as *familial-spatial data* and *familial-spatial correlations*, respectively. In the familial-spatial setup, we note that apart from the above-mentioned fixed covariates and vector of location random effects, the responses within each family might be affected by a common unobserved variable, say a *family random effect*. It is of interest to evaluate the effect of regression coefficients of the responses after exploiting familial-spatial correlations within and between families.

In the second part of this thesis, we extend the cluster-based spatial correlation idea of Mariathas and Sutradhar (2016) to develop models for spatial-temporal data by considering both spatial and temporal correlations between the linear responses. We refer to these correlations among the responses collected from different locations and different time points as *spatial-temporal correlations*. In addition to the fixed covariates and vector of location random effects that affect the responses, for this model there is an autocorrelation structure between the responses of different time points. In the following two sections, we review the spatial and spatial-temporal linear mixed effect models in the previous literature, respectively.

1.1 Spatial Data Models and Analysis

Several authors including Cressie (1993), Vecchia (1988, 1992), Jones and Vecchia (1993), Gaetan and Guyon (2010), Cressie and Johannesson (2008), Kang and Cressie (2011), and Mariathas and Sutradhar (2016) have published articles on the analysis of continuous spatial data and investigated their correlation structures.

For Gaussian spatial responses, Mardia and Marshall (1984) considered the linear model and determined the ML estimates of the model parameters. They assumed that the errors are correlated, and that their covariance follows a distribution with unknown parameters. Vecchia (1988), proposed a linear model with error terms that follow a second-order stationary Gaussian random process. He assumed that the responses contained noisy observations and collected from irregularly spaced locations. The ML estimates and model identification procedures are presented. Afterward, for the same location's pattern, Vecchia (1992) developed a new prediction methodology for the ML estimation of model parameters with the same error assumptions mentioned in his previous paper from 1988.

Cressie (1991, Chapter 2) considered different correlation structures of the spatial model such that they are a function of the Euclidian distance of two locations. To have a positive definite correlation function, Cressie (1991) has suggested using Gaussian and exponential covariance functions. He noted that by using any of these correlation structures, as the distance between two locations increases, the correlation tends to zero. Basu and Reinsel (1993, 1994) compared the generalized least square (GLS) and ordinary least square (OLS) estimators of linear regression model in a two-dimensional regular rectangular grid where errors are spatially correlated and follow an autoregressive moving average (ARMA) model. One of the drawbacks of fitting the ARMA model is that having responses from locations with equal distances is not always possible for a large spatial area. When responses are spatially correlated, Gelfand et al. (2003) considered the Gaussian stationary process model in a Bayesian framework.

Latent variables from neighbouring locations can have different effects on the response variable; therefore, the correlation between responses is highly dependent on the impact of these correlated latent variables. Significantly, the shorter the Euclidean distance between the locations, the higher the dependency between the responses. In the presence of correlated latent variables (random effect), fixed covariates, and the error term, pair-wise correlations between the spatial responses on a linear model have been obtained (Jones and Vecchia 1993; and Mariathas and Sutradhar 2016).

Assume in a study area there are S irregularly spaced locations. Let y_s be a continuous response measured at location $s = 1, 2, ..., S$, along with a vector of p-dimensional fixed covariates $x_s = (x_{s1}, x_{s2}, ..., x_{sp})'$. Let the unobservable effect of location s be denoted by γ_s^*. Jones and Vecchia (1993) were the first to consider a linear mixed model for the spatial data (x_s, y_s) given by

$$y_s = x_s'\beta + \gamma_s^* + \epsilon_s, \quad \text{for} \quad s{=}1,...,S \tag{1.1}$$

where $\beta = (\beta_1, \beta_2, ..., \beta_p)'$ is a vector of p-dimensional regression parameters, and ϵ_s is an error term that is identically and independently normally distributed with mean zero and variance of σ_ϵ^2. That is, $\epsilon_s \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$ for all $s = 1, 2, ..., S$. The model in matrix format can be written as

$$Y = X\beta + \gamma + \epsilon, \tag{1.2}$$

where Y is a vector of responses, X is a $S \times p$ design matrix, $\gamma = (\gamma_1^*, \gamma_2^*, ..., \gamma_S^*)'$ is a vector of random effects, and $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_S)'$ is a vector of independent observational errors with $cov(\epsilon) = \sigma_\epsilon^2 I_S$ in which I_S is an identity matrix. Each component of vector γ was assumed to follow a distribution with mean zero and a rational spectrum correlation structure with a covariance matrix of $cov(\gamma) = \sigma_\gamma^2 C$, where C is a $S \times S$ correlation matrix. It should be noted that the elements of γ and ϵ are independent. The total covariance of the model was given by

$$\sigma_\gamma^2 V = \sigma_\gamma^2 (C + \sigma_0^2 I_S) \tag{1.3}$$

where $\sigma_0^2 = \frac{\sigma_\epsilon^2}{\sigma_\gamma^2}$. Jones and Vecchia (1993) applied the ML method to estimate the regression parameters and variance of latent variables with a known matrix of V and normally distributed observational error as below

$$\hat{\beta} = (X'V^{-1}X)^{-1}(X'V^{-1}y) \tag{1.4}$$

$$\hat{\sigma}_{\gamma}^{2} = \frac{1}{S}(y - X\hat{\beta})'V^{-1}(y - X\hat{\beta}). \tag{1.5}$$

These ML estimates lead to having the GLS estimates. Since the distance between two locations, say d, has a substantial role, Jones and Vecchia (1993) utilized a modified Bessel function of the second kind, order 1, to construct elements of the covariance matrix V. Note that the elements of V are functions of σ_{γ}^{2}, σ_{ϵ}^{2}, ϕ, δ, and d, in which ϕ and δ are two additional parameters emerging from the partial differential equation (see equations (3) and (6) in Jones and Vecchia, 1993, page 948). One of the drawbacks of using a modified Bessel function is the complex and time-consuming computations of the elements of the covariance. Moreover, for the far apart locations, the correlation between two responses may approach zero very slowly compared to the exponential or Gaussian covariance functions presented by Cressie (1991, pages 85-87).

In two-dimensional coordinates, Basu and Reinsel (1994) proposed a regression model such that the errors are spatially correlated and follow a first order ARMA model. In a two-dimensional regular grid, the regression model is

$$y_{ij} = x_{ij}'\beta + \epsilon_{ij}, \quad i = 1, ..., m, \quad j = 1, ..., n, \tag{1.6}$$

where y_{ij} is the response variable at location (i, j), and x_{ij} is a p-dimensional vector of covariate corresponding to location (i, j). The model error, that is, ϵ_{ij} is given by

$$\epsilon_{ij} = \alpha_1\epsilon_{i-1,j} + \alpha_2\epsilon_{i,j-1} + \alpha_3\epsilon_{i-1,j-1} + \theta_1\eta_{i-1,j} + \theta_2\eta_{i,j-1} + \theta_3\eta_{i-1,j-1} + \eta_{ij}, \tag{1.7}$$

where $E(\eta_{ij}) = 0$ and $var(\eta_{ij}) = \sigma_\eta^2$ and for all i and j, and η_{ij}'s are independent, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)'$, $\theta = (\theta_1, \theta_2, \theta_3)'$ are model parameters. In matrix notation the model (1.6) can be written as,

$$Y = X\beta + \epsilon. \tag{1.8}$$

The GLS estimation of regression parameters has been compared with OLS estimation when errors are spatially correlated. The ML and restricted maximum likelihood (REML) estimation of $\xi = (\alpha', \theta', \sigma_\eta^2)'$ were also compared. Based on the parameter estimates and the magnitude of log-likelihood function, the non-separable spatial ARMA model was found to perform well compared to the separable or multiplicative ARMA model. It should be noted that this time-domain ARMA model performs well under the equally-spaced locations assumptions.

Mariathas and Sutradhar (2016) proposed a new approach by constructing a cluster/family around each location. More specifically, they assumed that a sequence of S locations are at equal distances from each other and that the distance between any two adjacent locations is one unit. The response at each location was assumed to be influenced by fixed covariates in addition to a vector of latent variables where the components of this vector are the random effects of the same and neighboring locations within a user-specified distance. The dimension of the vector of random effects may vary from one location to another and depends on a user-specified Euclidian distance value. The model was therefore referred to as an unbalanced random effects model. This user-specified distance is identified prior to the studies by the researchers. In the spatial set up, as the distance between the two locations increases, it is predicted that the correlation between the response variables resulting from these two locations approaches zero.. Mariathas and Sutradhar (2016) discussed the consequences of the random effects on the responses and modeled the pair-wise correlation structure of the spatial responses. At the s^{th} location, $s = 1, ..., S$, Mariathas and Sutradhar (2016) proposed the unbalanced linear mixed model given by

$$y_s = x_s'\beta + \omega_s'\tilde{\gamma}_s + \epsilon_s, \tag{1.9}$$

where $\beta = (\beta_1, \beta_2, ..., \beta_p)'$ is a p-dimensional vector of regression effects, and $\omega_s = (\omega_{s1}, ..., \omega_{sn_s})'$ is a known vector of weights that corresponds to $\tilde{\gamma}_s$. Let $\gamma = (\gamma_1^*, ..., \gamma_w^*, ..., \gamma_s^*, ..., \gamma_S^*)'$ be the vector of all random effects due to the S locations and $\tilde{\gamma}_s = (\gamma_{s1}, \gamma_{s2}, ..., \gamma_{sn_s})'$ be a vector of random effects due to the n_s locations within the cluster around location s. Thus, $\tilde{\gamma}_s \subset \gamma$. Note that n_s is the dimension of the vector $\tilde{\gamma}_s$, that is, the number of locations in the cluster around location s. Moreover, ϵ_s is an error term that is identically and independently distributed with mean zero and a variance of σ_ϵ^2. That is, $\epsilon_s \overset{iid}{\sim} (0, \sigma_\epsilon^2)$, $s = 1, 2, ..., S$.

As far as the distance between any two locations is concerned, Mariathas and Sutradhar (2016) assumed that d is a user-specified distance, and d_{ws} is the Euclidian distance between the w^{th} and the s^{th} locations. For both independent and correlated (equi-correlated) random effects, they proposed a linear mixed effect model and obtained the correlation structure between any two observations from the same and different clusters. The marginal distribution of each random effect was assumed to follow a normal distribution as,

$$\gamma_s^* \sim N(0, \sigma_\gamma^2), \tag{1.10}$$

with a pair-wise correlation structure given by

$$corr(\gamma_w^*, \gamma_s^*) = \delta_{ws}\phi_{ws}^* = \begin{cases} 1, & \text{for } d_{ws} = 0 \\ \phi, & \text{for } 0 < d_{ws} \leqslant d \\ 0, & \text{for } d_{ws} > d \end{cases} \tag{1.11}$$

where ϕ_{ws}^* is a correlation coefficient between the two random effects at the w^{th} and the s^{th} locations. In (1.11), the random effects within the user-specified distance d are assumed to be equi-correlated, and ϕ is an equi-correlation coefficient. Furthermore, δ_{ws} is an indicator variable defined by

$$\delta_{ws} = \begin{cases} 1, & \text{if } d_{ws} \leqslant d \;\; \text{for } w = 1, ..., S \\ 0, & \text{otherwise.} \end{cases} \tag{1.12}$$

Accordingly, in the case of independent random effects ($\phi = 0$), the pair-wise covariances of two responses have the form of

$$cov(y_w, y_s) = \Sigma = \begin{cases} \sigma_{ww}(\{\sigma_\gamma^2, \sigma_\epsilon^2\}), & \text{for } w = s, \; w = 1, ..., S \\ \sigma_{ws}(\{\sigma_\gamma^2\}|n_w, n_s, n_{ws}^*), & \text{for } w \neq s. \end{cases} \tag{1.13}$$

Also, in the case of equi-correlated random effects, the pair-wise covariances are given by

$$cov(y_w, y_s) = \Sigma = \begin{cases} \sigma_{ww}(\{\sigma_\gamma^2, \sigma_\epsilon^2, \phi\}|n_w), & \text{for } w = s, \; w = 1, ..., S \\ \sigma_{ws}(\{\sigma_\gamma^2, \phi\}|n_w, n_s, n_{ws}, n_{ws}^*, n_w^*, n_s^*), & \text{for } w \neq s \end{cases} \tag{1.14}$$

where Σ in (1.14) is a function of σ_γ^2, σ_ϵ^2, ϕ, and additional quantities of n_w, n_s, n_{ws}, n_{ws}^*, n_w^*, and n_s^* (number of location random effects) arising from decomposition of the adjacent clusters. After developing the spatial correlation model, Mariathas and Sutradhar (2016) examined the performance of MM and ML techniques to estimate regression effects, variance, and correlation parameters of the proposed model.

As far as this decomposition of the clusters is concerned, the construction of the correlation model for spatial counts and binary data is more complicated than for linear spatial data. Wijekoon et al. (2019) developed a correlation model for spatial count data. They investigated the application of the proposed model on lip cancer data (Clayton and Kaldor, 1987) collected from a sequence of spatial locations. Moreover, Sutradhar and Oyet (2020) recently accommodated the issue of constructing the correlation model of spatial binary responses by employing a mixed logistic model. It should be noted that a great deal of attention must be paid when modelling spatial-temporal correlations.

1.2 Spatial-Temporal Data Models and Analysis

At location s, $s = 1, 2, ..., S$, let y_{st} be an observation collected or measured at time point t, $t = 1, 2, ..., m$. The response y_{st} collected over both location s and time t is commonly referred to as spatial-temporal data. For example, the spatial-temporal response can be the amount of specific particulate in the air measured at different locations every hour. Several authors have analyzed continuous spatial-temporal data under a variety of assumptions. One of the common assumptions is the classes of the separable and non-separable space-time covariance function. According to Gaetan and

Guyon (2010), a space-time covariance function, $C(s,t)$, can be separable if:

$$\text{(a) additive:} \quad C(s,t) = C_S(s) + C_T(t); \tag{1.15}$$

$$\text{(b) factorizing:} \quad C(s,t) = C_S(s)C_T(t), \tag{1.16}$$

where $C_S(.)$ is a purely spatial covariance and $C_T(.)$ is a purely temporal covariance function. One of the separable model's benefits is the simplification of the calculation of the determinant and inverse of a spatial-temporal covariance matrix. Cressie and Huang (1999) considered a linear spatial-temporal model of wind speed measured every 6 hours at S locations. They assumed that the spatial-temporal response vector is $y = (y_{11}, ..., y_{st}, ..., y_{Sm})'$ with $\mu_{st} = E(Y_{st})$ and the covariance function between responses y_{st} and y_{rq} as,

$$K(s,r;t,q) = cov(y_{st}, y_{rq}). \tag{1.17}$$

It was assumed that the covariance function is stationary in space and time; that is,

$$K(s,r;t,q) = C(s-r;t-q), \tag{1.18}$$

for certain functions C with a positive-definiteness condition. Therefore, they introduced the classes of restricted non-separable covariance functions C, which depend only on the known Fourier integral.

Gneiting (2002) extended the approach of Cressie and Huang (1999) to a very general class of space-time covariance function and removed the earlier mentioned limitation relating to the closed-form Fourier integral.

Gaetan and Guyon (2010) considered a linear model for the spatial-temporal data

y_{st} given by

$$y_{st} = \alpha y_{(s-1)t} + \beta y_{s(t-1)} - \alpha\beta y_{(s-1)(t-1)} + \epsilon_{st}, \quad |\alpha| \text{ and } |\beta| < 1 \tag{1.19}$$

where $(1-\alpha B_1)(1-\beta B_2)y_{st} = \epsilon_{st}$. Note that B_1 and B_2 are the lag operators relative to coordinates s and t, respectively ($B_1 y_{st} = y_{(s-1)t}$, $B_2 y_{st} = y_{s(t-1)}$, and $B_1 B_2 y_{st} = y_{(s-1)(t-1)}$). They deduced that the covariance function of y is separable given by

$$C(s-s', t-t') = \sigma^2 \alpha^{|s-s'|} \beta^{|t-t'|}, \tag{1.20}$$

where $\sigma^2 = \sigma_\epsilon^2 (1-\alpha^2)^{-1}(1-\beta^2)^{-1}$.

Also, Fassó and Cameletti (2007) used the expectation-maximization (EM) algorithm to perform the ML estimation of a general three-stage linear spatial-temporal hierarchical random effects model. This hierarchical model has the form given by

$$Y_t = U_t + \epsilon_t, \tag{1.21}$$

$$U_t = X_t \beta + K\gamma_t + \omega_t, \tag{1.22}$$

$$\gamma_t = G\gamma_{t-1} + \eta_t, \tag{1.23}$$

where $Y_t = (y_{1t}, ..., y_{St})'$ is a vector of responses at time t for S spatial locations. Moreover, ϵ_t is a normal error term with mean zero and a variance-covariance matrix of $\sigma_\epsilon^2 I_S$, and U_t is a smoothed version of the spatial-temporal variable Y_t. In equation (1.22), let $X_t = (x_{1t}, ..., x_{St})'$ be an $S \times p$ design matrix, β be a p-dimensional vector of regression coefficient, γ_t denote a $q \times 1$ vector of unobservable time random effect with K as a $S \times q$ known weight matrix ($q \leqslant S$), and ω_t be the model error. In equation (1.23), the temporal dynamics of γ_t follows a q-dimensional autoregressive model with the $q \times q$ transition matrix G, and η_t is a q-dimensional vector of innovation error. It is assumed that the three error components ϵ_t, ω_t, and η_t are mutually independent such that

$$\epsilon_t \overset{iid}{\sim} N(0, \sigma_\epsilon^2 I_S), \tag{1.24}$$

$$\omega_t \overset{iid}{\sim} N(0, \sigma_\omega^2 C_\theta(h)), \tag{1.25}$$

$$\eta_t \overset{iid}{\sim} N(0, \Sigma_\eta), \tag{1.26}$$

where σ_ϵ^2 is a constant over time and space, I_S is an S-dimensional identity matrix, $\sigma_\omega^2 C_\theta(h)$ is a time-constant spatial covariance function with $h = ||s - s'||$ the Euclidean distance between two locations s and s', and Σ_η is a $q \times q$ matrix of variance-covariance of η_t. Previous studies have focused on the different classes of separable and non-separable space-time covariance functions and have not drawn enough attention for correlation between spatial locations.

One of the common challenges of these studies is how to deal with massive datasets and missing data. In the empirical-Bayesian framework, Kang et al. (2010) applied a fixed rank filtering (FRF) approach to reduce spatial and temporal dimension. Their spatial-temporal random effect (STRE) model with temporal dependency at a specific time point t is

$$\begin{aligned} y_{st} &= \mu_{st} + \nu_{st} + \epsilon_{st} \\ &= x'_{st}\beta_t + \omega'_{st}\gamma_t + \eta_{st} + \epsilon_{st}, \quad s = 1, ..., S, \quad t = 1, ..., m, \end{aligned} \tag{1.27}$$

where x_{st} is the p-dimensional fixed covariate at location s and time point t, ω_{st} is a q-dimensional vector of known spatial basis functions and $\gamma_t = (\gamma_{1t}, ..., \gamma_{qt})'$ is a q-dimensional vector of STRE with zero mean and the $q \times q$ variance-covariance matrix given by K_t. In general, the quantity $\mu_{st} = x'_{st}\beta_t$ is called a trend function, and $\nu_{st} = \omega'_{st}\gamma_t + \eta_{st}$ has zero mean and follows an STRE model. In the STRE model, it is assumed that γ_t, η_{st}, and ϵ_{st} are mutually independent with the following structures:

$$\gamma_t = G\gamma_{t-1} + \xi_t, \tag{1.28}$$

$$\eta_{st} \stackrel{iid}{\sim} (0, \sigma^2_\eta), \tag{1.29}$$

$$\epsilon_{st} \stackrel{iid}{\sim} (0, \sigma^2_\epsilon). \tag{1.30}$$

In equation (1.28), γ_t is a vector autoregressive (VAR) process where G is the $q \times q$ first-order autoregressive matrix and ξ_t the q-dimensional innovation vector with mean zero and $var(\xi_t) = U_t$. A subject that was not discussed in this paper was the dimension selection of the vector of random effect γ_t at each location as well as the selection of known components of the vector of ω_{st}. It was assumed that there is an equal number of random effects that affect the response at each location, where in practice that is not always the case. Therefore, for both the SRE and STRE models, it is noteworthy to find a suitable scheme for the dimension of vector of random effects at each location.

1.3 Motivation and Contribution to the Literature

Previous studies discussed in Section 1.1 have been limited to linear models with a single observation obtained from each location while modelling the spatial correlation between the responses. Jones and Vecchia (1993) and Basu and Reinsel (1994) proposed linear models such that the response at each location was influenced by one location random effect from that location. Mariathas and Sutradhar (2016) extended this model by assuming there is a cluster around each location. This cluster surrounds nearby locations, and the response at each location was influenced by certain location random effects within the cluster at a user-specified distance. No existing literature has considered the linear models with multivariate observations at each location. Therefore, it seems to be more challenging to assess the linear responses that are correlated in two ways; spatial correlations happen due to the location random effects from nearby locations, and familial correlations occur due to a shared family random effect. Hence, one of the challenges of this model is formulating and calculating the correlation of responses within and between families due these correlations. We noticed that the analysis of such continuous familial-spatial data and familial-spatial correlation structure have not been studied in the literature. Thus, the complexity in modelling the correlation between the responses motivates us to investigate this situation as it has an application in epidemiology and environmental studies. In addition, most studies in Section 1.2 have tended to focus on classes of separable and non-separable covariance functions for spatial-temporal models. Also, in the analysis of spatial-temporal linear models, none of these studies considered a cluster of locations. Hence, in light of recent location classification schemes, it is of interest to see the effects of the fixed covariates and the location random effects on the linear responses that follow an autocorrelation structure. The results in the familial-spatial models motivate us to investigate the impact of location random effects on the repeated responses over time. Therefore, we derived the cluster-based spatial-temporal correlations between the responses of the same and different locations at the same and different time points.

In Chapter 2 of this thesis, we extend the spatial linear mixed model of Mariathas and Sutradhar (2016) to the familial-spatial linear mixed model, where a group of random effects from adjacent locations (say location random effects) and a common unobservable variable for all family members of a location (say family random effects) are used to model the cluster-based familial-spatial correlations of responses between and within the families.

In Chapter 3, we examine the performance of the proposed model in Chapter 2 by using the well-known GLS and MM techniques for both independent and correlated location random effects models along with several simulation studies.

Chapter 4 deals with a linear dynamic mixed model for spatial-temporal data. We implement the analysis of continuous data with weighted location random effects through an autoregressive model of order one and obtain the cluster-based spatial-temporal correlations between responses. A marginal generalized quasi-likelihood (GQL) estimation approach is used to estimate the regression coefficients, spatial correlation of location random effects, and variance of location random effects. To estimate the dynamic parameter and variance of the model, we carry out the MM approach.

Finally, Chapter 5 summarizes how the proposed models are developed, and we discuss possible future extensions.

Chapter 2

Familial-Spatial Linear Mixed Models for Multivariate Data

For spatial Gaussian data, Mariathas and Sutradhar (2016) and Jones and Vecchia (1993) considered models with a single observation at each location. In this chapter, we propose an extension to familial-spatial random effect regression models when there is a family of observations at each location. For example, a researcher may be interested in evaluating the effects of individual, epidemiological, and environmental covariates such as age, gender, and water hardness on the blood lead level (BLL) of residents. In this case, aside from the correlation between observations measured at nearby locations, the family of observations at a given location are also likely to be correlated. We also assume that the familial correlation at each location is induced by a common unobservable family random effect. These family random effects at each location are independent and do not affect the responses of neighbouring locations. In Section 2.4, we provide the marginal and correlation properties of the proposed model.

2.1 Proposed Familial-Spatial Linear Mixed Effect Model

The existing model developed by Mariathas and Sutradhar (2016) described the structure of a region $\mathcal{S}$ with a sequence of S spatial locations such that there is only one response at each location. In the familial-spatial setup, we propose a new linear mixed effect model for a family, or vector of observations measured at each location along with covariate information. As we mentioned earlier in Chapter 1, this group of observations at a certain location constitutes a family, and locations within a user-specified distance constitute a cluster.

Let y_{si} be the response on a continuous scale for the i^{th} $(i = 1, ..., m)$ family member at the s^{th} $(s = 1, ..., S)$ location. Each of these mS responses might be influenced by a p-dimensional environmental or individual fixed covariate vector $x_{si} = (x_{si1}, ..., x_{sip})'$. Apart from this fixed vector of information, we also assumed that the response is influenced by a vector of location random effects such that this vector contains unobservable random effects from itself and nearby locations. As an example, let y_{si} be the value of damages to homes caused by a storm. Clearly, damages to homes in the same neighborhood will be about the same. Thus, these homes will form a cluster. Neighborhoods that are a certain distance apart may affect each other due to certain unobservable random effects. Let C_s be a cluster of random effects centred around the location s, containing locations that are correlated, denoted by $\tilde{\gamma}_s = (\gamma_{s1}, ..., \gamma_{sj}, ..., \gamma_{sn_s})'$ for all $s = 1, 2, ..., S$. Furthermore, a response at the s^{th} location may be affected by a family random effect that is common to all responses at the s^{th} location. This family random effect is denoted by α_s, for $s = 1, 2, ..., S$. Figure 2.1 (see page 21) clearly presents the structure of the locations, location random effects, family random effects, and responses. It can be seen that at each location there is only one location random effect γ_s^*, and all responses at a particular location will be affected by a family random effect α_s. Let the vector of all S location random effects be denoted by $\gamma = (\gamma_1^*, \gamma_2^*, ..., \gamma_S^*)'$, and each component of this vector has a distribution with mean zero and variance of σ_γ^2.

Following Mariathas and Sutradhar (2016), let n_w be the number of locations in the cluster centred around location w (C_w), that is, the cluster size, and d_{wr} be the Euclidian distance between locations w and r. Naturally, as the linear distance between locations increases, the correlation between latent variables of these locations decreases. Therefore, it is important to know how far the locations are from each other to make the latent variables uncorrelated. This user-specified distance is denoted by d, which is selected by the researcher. Define the indicator variable δ_{wr} as,

$$\delta_{wr} = \begin{cases} 1, & \text{if } d_{wr} \leqslant d \quad \text{for} \quad r\text{=1,..., } S \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

then

$$\sum_{r \in C_w} \delta_{wr} = n_w. \tag{2.2}$$

It is noted that the larger the value of d, the greater the cluster size. Now, based on the value of d, it is assumed that any two locations whose distance is less than or equal to d have correlated location random effects, and those with a distance greater than d have uncorrelated location random effects, hence:

$$corr(\gamma_w^*, \gamma_s^*) = \delta_{ws}\phi_{ws}^*. \tag{2.3}$$

For the convenience of modelling the correlation structure and calculations, suppose that any two locations within distance d have equi-correlated location random effects as,

$$corr(\gamma_w^*, \gamma_s^*) = \begin{cases} 1, & \text{for } d_{ws} = 0 \\ \phi, & \text{for } 0 < d_{ws} \leqslant d \\ 0, & \text{for } d_{ws} > d. \end{cases} \tag{2.4}$$

Also, suppose that a family of observations at each location is affected by a common family random effect, which is denoted by α_s. It then follows that

$$\alpha_s \overset{iid}{\sim} (0, \sigma_\alpha^2). \tag{2.5}$$

Then, our proposed linear mixed model for familial-spatial data is defined as,

$$y_{si} = x'_{si}\beta + \omega'_s\tilde{\gamma}_s + \alpha_s + \epsilon_{si}, \quad s = 1, 2, ..., S; \quad i = 1, 2, ..., m \tag{2.6}$$

where β is a $p \times 1$ regression effect of x_{si} covariates, $\omega_s = (\omega_{s1}, \omega_{s2}, ..., \omega_{sn_s})'$ is the $n_s \times 1$ known weight vector corresponding to $\tilde{\gamma}_s \in C_s$, which is selected by the researcher, and $\alpha_s \overset{iid}{\sim} (0, \sigma_\alpha^2)$ is a family random effect, which is defined in (2.5). In (2.6), ϵ_{si} denotes the error term for the i^{th} member of the s^{th} location, and it follows that

$$\epsilon_{si} \overset{iid}{\sim} (0, \sigma_\epsilon^2). \tag{2.7}$$

Note that although the proposed model in (2.6) and the model considered by Mariathas and Sutradhar (2016, equation (2.6), page 5) look similar, they are fundamentally different. The difference between the two models is that, in Mariathas and Sutradhar's model, there is only one observation at each location, whereas in (2.6), we observe a family of observations at each location. Moreover, these observations at each location are affected by a common family random effect α_s. The following diagram represents the locations and family structure.

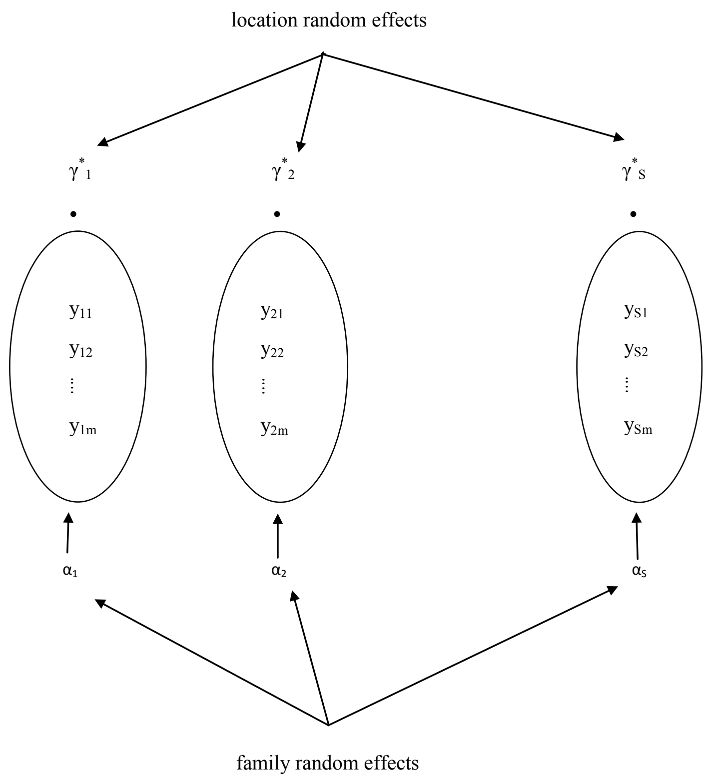

Figure 2.1: Structure of familial data at each location, correlated location random effects γ_s^* for $s = 1, 2, ..., S$, and independent family random effects α_s for $s = 1, 2, ..., S$.

2.2 Decomposition of Cluster Regions

As previously stated, the location random effects in each cluster are correlated, and, subsequently, the responses within and between clusters are likely to be correlated. Specifically, y_{si} is influenced by n_s location random effects such that $n_s - 1$ of them are from adjacent locations but they all belong to a cluster around the s^{th} location

(C_s), and they are denoted by $\tilde{\gamma}_s = (\gamma_{s1}, ..., \gamma_{sj}, ..., \gamma_{sn_s})'$. To facilitate the derivation of the basic properties of the familial-spatial responses, it is necessary to distinguish between the different regions of two neighbouring clusters and specify the number of location random effects in each region. We consider another response from the location w denoted by $y_{wi'}$ $(s \neq w)$. This response will be influenced by n_w random effects within the cluster C_w as $\tilde{\gamma}_w = (\gamma_{w1}, ..., \gamma_{wj}, ..., \gamma_{wn_w})'$. Based on the magnitude of d_{ws}, two clusters of C_w and C_s may have common locations. Thus, $y_{wi'}$ and y_{si} are affected by the same random effects from common locations. Moreover, there might be some random effects that belong to only C_w and are not in C_s, while some other random effects may belong to only C_s and not in C_w such that they are correlated. For the purpose of constructing the correlation structure of $y_{wi'}$ and y_{si}, each cluster is classified into three regions as follows:

- The shared area between C_w and C_s denoted by ws, where n^*_{ws} is the number of common locations (or location random effects) between the clusters C_w and C_s.

- The area in C_w that has no common location with C_s denoted by $w(1) \cup w(2)$, and similarly, the area in C_s that has no common location with C_w denoted by $s(1) \cup s(2)$. Let $n^*_{w(2)}$ be the number of random effects in C_w and not in C_s that are correlated with $n^*_{s(2)}$ random effects in C_s. It should be noted that not all of these uncommon pairs are correlated. Hence, the number of those correlated uncommon pairs of random effects is denoted by n_{ws}.

- We denote $n^*_{w(1)}$ by the number of random effects in C_w that are not correlated with any of $n^*_{s(1)} + n^*_{s(2)}$ random effects in C_s. Similarly, $n^*_{s(1)}$ is the number of random effects in C_s that are not correlated with $n^*_{w(1)} + n^*_{w(2)}$ random effects in C_w.

Figure 2.2 shows the decomposition of two adjacent clusters. This decomposition leads to expressing n_w and n_s as,

$$n_w = n^*_{w(1)} + n^*_{w(2)} + n^*_{ws} \quad \text{and} \quad n_s = n^*_{s(1)} + n^*_{s(2)} + n^*_{ws}. \tag{2.8}$$

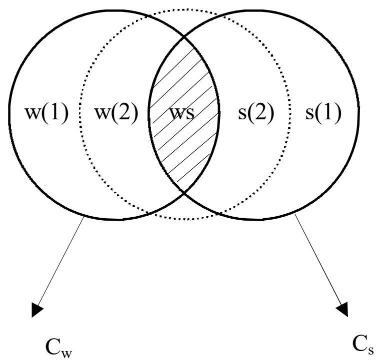

Figure 2.2: A graphical decomposition of two clusters of C_w and C_s.

For two different locations w and s, the vectors of location random effects belonging to clusters C_w and C_s are $\tilde{\gamma}_w = (\gamma_{w1}, \gamma_{w2}, ..., \gamma_{wn_w})'$ and $\tilde{\gamma}_s = (\gamma_{s1}, \gamma_{s2}, ..., \gamma_{sn_s})'$, respectively. Based on the above decomposition, the number of locations that only belonging to C_w and not C_s is denoted by n^*_w ($n^*_w = n^*_{w(1)} + n^*_{w(2)}$), and similarly, the number of locations that only belonging to C_s and not C_w is denoted by n^*_s ($n^*_s = n^*_{s(1)} + n^*_{s(2)}$). Also, one can express n_w and n_s as,

$$n_w = n^*_w + n^*_{ws} \quad \text{and} \quad n_s = n^*_s + n^*_{ws}. \tag{2.9}$$

Following the above decomposition, the vectors of location random effects $\tilde{\gamma}_w$ and $\tilde{\gamma}_s$ can also be decomposed as

$$\tilde{\gamma}_w = (\tilde{\gamma}'_{w(1)}, \tilde{\gamma}'_{w(2)}, \tilde{\gamma}'_{ws})' \tag{2.10}$$

and

$$\tilde{\gamma}_s = (\tilde{\gamma}'_{ws}, \tilde{\gamma}'_{s(2)}, \tilde{\gamma}'_{s(1)})', \tag{2.11}$$

respectively. The corresponding weight vectors ω_w and ω_s can be decomposed as,

$$\begin{aligned}\omega_w =&(\omega_{w1}, ..., \omega_{wj}, ..., \omega_{wn_w})' \\ =&(\omega'_{w(1)}, \omega'_{w(2)}, \omega'_{ws})'\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\omega_s =&(\omega_{s1}, ..., \omega_{sj}, ..., \omega_{sn_s})' \\ =&(\omega'_{ws}, \omega'_{s(2)}, \omega'_{s(1)})',\end{aligned} \tag{2.13}$$

respectively.

Mariathas and Sutradhar (2016) suggested using a known vector so that it gives identical weight to all location random effects belonging to C_w as,

$$\begin{aligned}\omega_w =&(\omega_{w1}, ..., \omega_{wj}, ..., \omega_{wn_w})' \\ =&(\frac{1}{\sqrt{n_w}}1'_{n^*_{w(1)}}, \frac{1}{\sqrt{n_w}}1'_{n^*_{w(2)}}, \frac{1}{\sqrt{n_w}}1'_{n^*_{ws}})' \\ =&\frac{1}{\sqrt{n_w}}1_{n_w},\end{aligned} \tag{2.14}$$

where $1_{n_w} = (1, 1, ..., 1)'$ is the $n_w \times 1$ unit vector. Our proposed mixed model can be written as,

$$y_{wi} = x'_{wi}\beta + \frac{1}{\sqrt{n_w}}1'_{n_w}\tilde{\gamma}_w + \alpha_w + \epsilon_{wi}, \quad w = 1, 2, ..., S; \quad i = 1, 2, ..., m. \tag{2.15}$$

When the location random effects are independent with $\gamma_{wj} \overset{iid}{\sim} (0, \sigma^2_\gamma)$, the variance of $\frac{1}{\sqrt{n_w}}1'_{n_w}\tilde{\gamma}_w$ is the same as the variance of each vector member. It satisfies

$$E\Big(\frac{1}{\sqrt{n_w}}1'_{n_w}\tilde{\gamma}_w\Big) = 0, \tag{2.16}$$

and

$$\begin{aligned} var\Big(\frac{1}{\sqrt{n_w}}1'_{n_w}\tilde{\gamma}_w\Big) &= var\Big(\frac{1}{\sqrt{n_w}}\sum_{j=1}^{n_w}\gamma_{wj}\Big) \\ &= \frac{1}{n_w}(n_w\sigma^2_\gamma) = \sigma^2_\gamma. \end{aligned} \tag{2.17}$$

One can also use the additive model with the weight vector $\omega'_w\tilde{\gamma}_w = 1'_{n_w}\tilde{\gamma}_w$, to understand the influence of neighboring location random effects. For the independent location random effects, the variance of this combination depends on the value of n_w, which means the wider the cluster, the larger the variance. Thus, we have

$$var(1'_{n_w}\tilde{\gamma}_w) = n_w\sigma^2_\gamma, \tag{2.18}$$

therefore, if $n_w \to \infty$, the variance in (2.18) approaches infinity. Another suggestion might be to consider the average of all location random effects in C_w, that is, $\omega'_w\tilde{\gamma}_w = \frac{1}{n_w}1'_{n_w}\tilde{\gamma}_w$. Similar to the additive model, for the independent location random effects, the variance of this combination depends on the value of n_w. In this case, the larger the cluster size, the smaller the variance. This implies that

$$var(\frac{1}{n_w}1'_{n_w}\tilde{\gamma}_w) = \frac{1}{n_w}\sigma^2_\gamma, \tag{2.19}$$

therefore, if $n_w \to \infty$, the variance in (2.19) approaches zero. In fact, both the unit weight and average weight models are not appropriate since they yield infinity and zero variances for the large cluster sizes, respectively.

According to Mariathas and Sutradhar (2016), another example of the weight vectors such that $var(\omega_w' \tilde{\gamma}_w) = \sigma_\gamma^2$ $(\omega_w' \omega_w = 1)$ is the exponential distance decaying weights given as,

$$\begin{aligned}\omega_w =& (\omega_{w1}, ..., \omega_{wj}, ..., \omega_{wn_w})' \\ =& (a^1 k, ..., a^j k, ..., a^{n_w} k)',\end{aligned} \tag{2.20}$$

where a is an appropriate constant function, and $k = \Big(\sum_{j=1}^{n_w} a^{2j} \Big)^{-0.5}$. On the other hand, for the correlated location random effects γ_{wj} and $\gamma_{wj'}$ with $corr(\gamma_{wj}, \gamma_{wj'}) = \phi_{jj'}(w)$, we deduce

$$\begin{aligned} var\Big(\frac{1}{\sqrt{n_w}} 1_{n_w}' \tilde{\gamma}_w \Big) &= \frac{1}{n_w} \Big(\sum_{j=1}^{n_w} var(\gamma_{wj}) + 2 \sum_{j<j'}^{n_w} cov(\gamma_{wj}, \gamma_{wj'}) \Big) \\ &= \frac{1}{n_w} \Big(n_w \sigma_\gamma^2 + 2\sigma_\gamma^2 \sum_{j<j'}^{n_w} \phi_{jj'}(w) \Big) \\ &= \sigma_\gamma^2 \big[1 + \frac{2}{n_w} \sum_{j<j'}^{n_w} \phi_{jj'}(w) \big]. \end{aligned} \tag{2.21}$$

As a spacial case, when location random effects belonging to a cluster are equi-correlated with $\phi_{jj'}(w) = \phi$, the variance in (2.21) can be simplified to

$$var\Big(\frac{1}{\sqrt{n_w}} 1_{n_w}' \tilde{\gamma}_w \Big) = \sigma_\gamma^2 [1 + (n_w - 1)\phi]. \tag{2.22}$$

Using (2.16), (2.17), and (2.22), it can be shown that the marginal mean and variance of y_{wi} are given as,

$$E(Y_{wi}) = \mu_{wi} = x_{wi}' \beta, \tag{2.23}$$

and

$$var(Y_{wi}) = \begin{cases} \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\epsilon^2, & \text{for } \gamma_{wj} \overset{iid}{\sim} (0, \sigma_\gamma^2) \\ \sigma_\gamma^2[1 + (n_w - 1)\phi] + \sigma_\alpha^2 + \sigma_\epsilon^2, & \text{for } \gamma_{wj} \sim (0, \sigma_\gamma^2), \text{ corr}(\gamma_{wj}, \gamma_{wj'}) = \phi, \end{cases} \tag{2.24}$$

respectively.

2.3 Computation and Comparison of n_w, n^*_{ws}, and n_{ws}

2.3.1 The Number of Locations (n_w) in Cluster C_w

In a linear sequence of spatial locations with a total number of S locations, the first location on the straight line from the left is labeled with $w = 1$. Also, the furthest location to the first location is labeled with $w = S$. By starting from the first location, that is, $w = 1$, and moving to the right and location $w = S$, the nearest location is $w = 2$, and the second nearest location is $w = 3$. We continue eventually getting to the last location (furthest location to $w = 1$), that is, $w = S$. Irrespective of the user-specified distance valued d, each location is 1 unit away from the nearest location. For example, the locations $w = 5$ and $w = 6$, and $w = 4$ and $w = 5$ are one unit apart. To find n_w, we first construct a circular region C_w centred at location w, with diameter d. Note that location $w = 1$ and location $w = S$ are at the boundaries of the region of interest. So, when $d = 2$, we have $n_1 = 2$, $n_2 = n_3 = ... = n_{S-1} = 3$, and $n_S = 2$. Values of n_w corresponding to $d = 2, 4$, and 6 are shown in Table 2.1.

Table 2.1: Number of locations (n_w) in cluster C_w for $d = 2$, 4, and 6.

	n_1	n_2	n_3	n_4	...	n_{S-3}	n_{S-2}	n_{S-1}	n_S
$d = 2$	2	3	3	3	...	3	3	3	2
$d = 4$	3	4	5	5	...	5	5	4	3
$d = 6$	4	5	6	7	...	7	6	5	4

In Table 2.2, we outline the vector of the location random effect and family random effect associated with each location when $S = 100$ and $d = 6$. For example, the number of locations or, equivalently, the number of location random effects in the cluster around the 6^{th} location (C_6) is $n_6 = 7$. The vector of the location random effect is denoted by $\tilde{\gamma}_6 = (\gamma_3^*, \gamma_4^*, \gamma_5^*, \gamma_6^*, \gamma_7^*, \gamma_8^*, \gamma_9^*)' = (\gamma_{61}, \gamma_{62}, \gamma_{63}, \gamma_{64}, \gamma_{65}, \gamma_{66}, \gamma_{67})'$.

Table 2.2: Family Random Effects and Location Random effects centred around each spatial location $w = 1, 2, ..., 100$ with $d = 6$.

Location	Cluster	Random Effect Vector	Family Random Effect
1	C_1	$\tilde{\gamma}_1 = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)'$ $\tilde{\gamma}_1 = (\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14})'$	α_1
2	C_2	$\tilde{\gamma}_2 = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*, \gamma_5^*)'$ $\tilde{\gamma}_2 = (\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25})'$	α_2
3	C_3	$\tilde{\gamma}_3 = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*, \gamma_5^*, \gamma_6^*)'$ $\tilde{\gamma}_3 = (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}, \gamma_{35}, \gamma_{36})'$	α_3
w	C_w	$\tilde{\gamma}_w = (\gamma_{w-3}^*, \gamma_{w-2}^*, \gamma_{w-1}^*, \gamma_w^*, \gamma_{w+1}^*, \gamma_{w+2}^*, \gamma_{w+3}^*)'$ $\tilde{\gamma}_w = (\gamma_{w1}, \gamma_{w2}, \gamma_{w3}, \gamma_{w4}, \gamma_{w5}, \gamma_{w6}, \gamma_{w7})'$	α_w
98	C_{98}	$\tilde{\gamma}_{98} = (\gamma_{95}^*, \gamma_{96}^*, \gamma_{97}^*, \gamma_{98}^*, \gamma_{99}^*, \gamma_{100}^*)'$ $\tilde{\gamma}_{98} = (\gamma_{98,1}, \gamma_{98,2}, \gamma_{98,3}, \gamma_{98,4}, \gamma_{98,5}, \gamma_{98,6})'$	α_{98}
99	C_{99}	$\tilde{\gamma}_{99} = (\gamma_{96}^*, \gamma_{97}^*, \gamma_{98}^*, \gamma_{99}^*, \gamma_{100}^*)'$ $\tilde{\gamma}_{99} = (\gamma_{99,1}, \gamma_{99,2}, \gamma_{99,3}, \gamma_{99,4}, \gamma_{99,5})'$	α_{99}
100	C_{100}	$\tilde{\gamma}_{100} = (\gamma_{97}^*, \gamma_{98}^*, \gamma_{99}^*, \gamma_{100}^*)'$ $\tilde{\gamma}_{100} = (\gamma_{100,1}, \gamma_{100,2}, \gamma_{100,3}, \gamma_{100,4})'$	α_{100}

2.3.2 The Number of Common Locations (n_{ws}^*) Between Clusters C_w and C_s

Let us assume w and s are two different locations in a straight line such that $s > w$, and the distance between them is $d_{ws} = |s - w|$. Figures 2.3, 2.4, and 2.5 exhibit the number of locations in each cluster and the number of common locations between any two clusters for three distinct values of $d = 2$, $d = 4$, and $d = 6$, respectively. Following each figure, there is a table that demonstrates the number of shared locations in two clusters of C_w and C_s. The number of common locations, n^*_{ws}, are given in Tables 2.3, 2.4, and 2.5 for $d = 2$, $d = 4$, and $d = 6$, respectively.

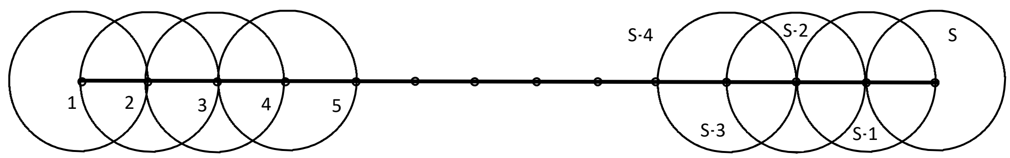

Figure 2.3: Graphical display of the cluster around each location with values of n_w and n^*_{ws} for $d = 2$.

Table 2.3: Number of common locations (n^*_{ws}) for $d = 2$ and $s > w$.

$\|s-w\|$	1	2	$\geqslant 3$
n^*_{ws}	2	1	0

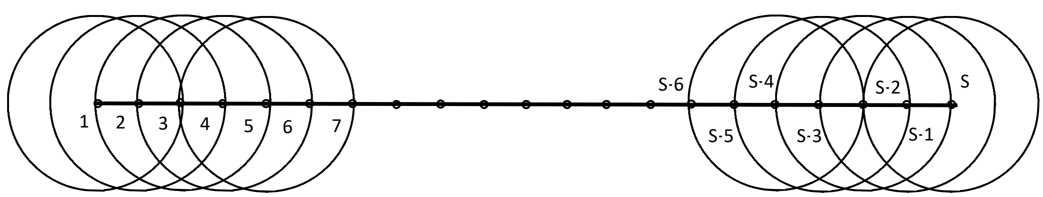

Figure 2.4: Graphical display of the cluster around each location with values of n_w and n^*_{ws} for $d = 4$.

Table 2.4: Number of common locations (n^*_{ws}) for $d = 4$ and $s > w$.

	$\|s-w\|$	1	2	3	4	$\geqslant 5$
(if $w = 1$ or $w = S-1$)	n^*_{ws}	3	3	2	1	0
(if $w = 2, 3, ..., S-2$)	n^*_{ws}	4	3	2	1	0

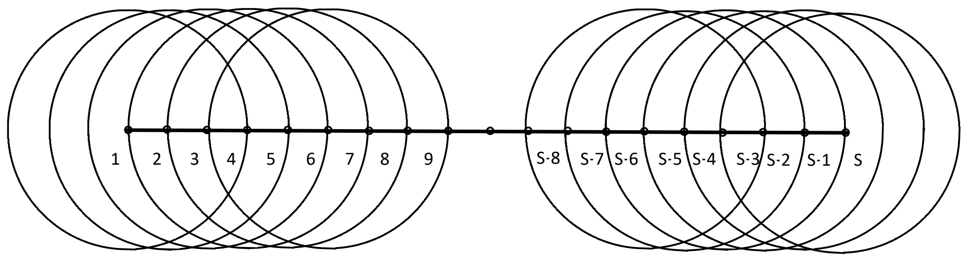

Figure 2.5: Graphical display of the cluster around each location with values of n_w and n^*_{ws} for $d = 6$.

Table 2.5: Number of common locations (n^*_{ws}) for $d = 6$ and $s > w$.

	$\lvert s-w \rvert$	1	2	3	4	5	6	$\geqslant 7$
(if $w = 1$)	n^*_{ws}	4	4	4	3	2	1	0
(if $w = 2$)	n^*_{ws}	5	5	4	3	2	1	0
(if $w = 3, 4, ..., S-3$)	n^*_{ws}	6	5	4	3	2	1	0
(if $w = S-2$)	n^*_{ws}	5	4	–	–	–	–	–
(if $w = S-1$)	n^*_{ws}	4	–	–	–	–	–	–

2.3.3 Number of Correlated Uncommon Pairs of Locations (n_{ws}) for Cluster C_w and C_s

In this section, the values of n_{ws}, $n^*_{w(2)}$, and $n^*_{s(2)}$ for selected pairs of (w, s), and for $d = 2$, $d = 4$, and $d = 6$ are given in Tables 2.6, 2.7, and 2.8, respectively. More specifically, Wijekoon et al. (2019) obtained a specific pattern between the value of n_{ws} and the known values of $n^*_{w(2)}$ and $n^*_{s(2)}$ when the distance is $d = 4$.

Table 2.6: Values of $n^*_{w(2)}$, $n^*_{s(2)}$, and n_{ws} for selected pairs of locations with $S = 100$ and $d = 2$.

(w, s)	$\|s - w\|$	$n^*_{w(2)}$	$n^*_{s(2)}$	n_{ws}
(1,2)	1	0	0	0
(1,3)	2	1	1	1
(1,4)	3	2	2	3
(1,5)	4	1	1	1
(1,6)	5	0	0	0
(2,3)	1	0	0	0
(2,4)	2	1	1	1
(2,5)	3	2	2	3
(2,6)	4	1	1	1
(3,7)	4	1	1	1
(3,10)	7	0	0	0
(97,100)	3	2	2	3
(98,99)	1	0	0	0
(98,100)	2	1	1	1
(99,100)	1	0	0	0

As can be seen from Table 2.6, for all possible pairs of (w, s), the values of $n^*_{w(2)}$ and $n^*_{s(2)}$ are equal; therefore by assuming $n^*_{w(2)} = n^*_{s(2)} = p$, we demonstrate that

$$n_{ws} = \frac{p(p+1)}{2}. \tag{2.25}$$

Table 2.7: Values of $n^*_{w(2)}$, $n^*_{s(2)}$, and n_{ws} for selected pairs of locations with $S = 100$ and $d = 4$.

(w, s)	$\|s-w\|$	$n^*_{w(2)}$	$n^*_{s(2)}$	n_{ws}
(1,2)	1	0	1	0
(1,3)	2	0	1	0
(1,4)	3	1	2	2
(1,5)	4	2	3	5
(1,6)	5	0	0	0
(2,3)	1	0	0	0
(2,4)	2	1	1	1
(2,5)	3	2	2	3
(2,6)	4	3	3	6
(3,7)	4	3	3	6
(3,8)	5	4	4	10
(3,10)	7	2	2	3
(97,100)	3	2	1	2
(98,99)	1	0	0	0
(98,100)	2	1	0	0
(99,100)	1	1	0	0

From Table 2.7, Wijekoon et al. (2019) found that by assuming $q = max(n^*_{w(2)}, n^*_{s(2)})$, then n_{ws} be expressed as,

$$n_{ws} = \begin{cases} 0, & \text{if } \; min(n^*_{w(2)}, n^*_{s(2)}) = 0 \\ \frac{(q+2)(q-1)}{2}, & \text{if } \; n^*_{w(2)} \neq n^*_{s(2)} \neq 0 \\ \frac{p(p+1)}{2}, & \text{if } \; n^*_{w(2)} = n^*_{s(2)} = p. \end{cases} \tag{2.26}$$

Table 2.8: Values of $n^*_{w(2)}$, $n^*_{s(2)}$, and n_{ws} for selected pairs of locations with $S = 100$ and $d = 6$.

(w, s)	$\|s - w\|$	$n^*_{w(2)}$	$n^*_{s(2)}$	n_{ws}
(1,2)	1	0	1	0
(1,3)	2	0	2	0
(1,4)	3	0	2	0
(1,5)	4	1	3	3
(1,6)	5	2	4	7
(2,3)	1	0	1	0
(2,4)	2	0	1	0
(2,5)	3	1	2	2
(2,6)	4	2	3	5
(3,7)	4	3	3	6
(3,8)	5	4	4	10
(3,10)	7	6	6	21
(4,5)	1	0	0	0
(96,100)	4	3	1	3
(97,100)	3	2	0	0
(98,99)	1	1	0	0
(98,100)	2	2	0	0
(99,100)	1	1	0	0

From the pattern observed in Table 2.8, we obtain n_{ws} as follows,

$$n_{ws} = \begin{cases} 0, & \text{if } \ min(n^*_{w(2)}, n^*_{s(2)}) = 0 \\ \frac{(q+2)(q-1)}{2}, & \text{if } \ n^*_{w(2)} \neq n^*_{s(2)} \neq 0, \ |n^*_{w(2)} - n^*_{s(2)}| = 1 \\ \frac{(q-2)(q+3)}{2}, & \text{if } \ n^*_{w(2)} \neq n^*_{s(2)} \neq 0, \ |n^*_{w(2)} - n^*_{s(2)}| = 2 \\ \frac{p(p+1)}{2}, & \text{if } \ n^*_{w(2)} = n^*_{s(2)} = p. \end{cases} \tag{2.27}$$

2.3.4 Examples for $d = 6$

Generally, in the linear sequence of locations with the user-specified distance equal to 6, when the distance of the centre of clusters C_w and C_s is greater than 6, there is no common location random effect.

Example 2.3.1. In this example, we consider one of the situations where $s - w > 6$. Figure 2.6 shows one of the cases where $s - w = 8$ units. Here, the distance of the centre of clusters C_w and C_s is $s - w = 8$. The number of locations belonging to each cluster is $n_w = n_s = 7$. Also, the number of common locations between the two clusters is $n^*_{ws} = 0$. In this spatial design, we count the number of correlated uncommon pairs of locations for two clusters. We select one location from each of these two clusters. The random effects from these two locations are correlated if the distance of these locations is less than or equal to $d = 6$. For instance, pairs of $(w + 3, s - 3)$ have a distance of $s - 3 - (w + 3) = 2$ units. Therefore, these two locations have correlated random effects.

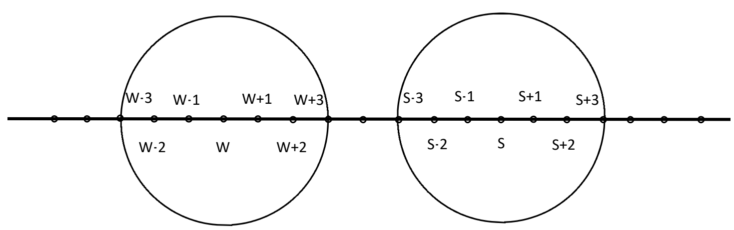

Figure 2.6: Graphical structure of two clusters with no common location random effects.

Table 2.9, shows how to calculate the number of correlated uncommon pairs of location random effects.

Table 2.9: Correlated uncommon pairs of location random effects.

Locations in C_w not in C_s	Locations in C_s not in C_w	Distance of Locations	Correlated Pairs (Yes/No)
$w-3$	$s-3$	$s-3-(w-3)=8>6$	No
	$s-2$	$s-2-(w-3)=9>6$	No
	$s-1$	$s-1-(w-3)=10>6$	No
	s	$s-(w-3)=11>6$	No
	$s+1$	$s+1-(w-3)=12>6$	No
	$s+2$	$s+2-(w-3)=13>6$	No
	$s+3$	$s+3-(w-2)=14>6$	No
$w-2$	$s-3$	$s-3-(w-2)=7>6$	No
	$s-2$	$s-2-(w-2)=8>6$	No
	$s-1$	$s-1-(w-2)=9>6$	No
	s	$s-(w-2)=10>6$	No
	$s+1$	$s+1-(w-2)=11>6$	No
	$s+2$	$s+2-(w-2)=12>6$	No
	$s+3$	$s+3-(w-2)=13>6$	No
$w-1$	$s-3$	$s-3-(w-1)=6\geqslant 6$	Yes
	$s-2$	$s-2-(w-1)=7>6$	No
	$s-1$	$s-1-(w-1)=8>6$	No
	s	$s-(w-1)=9>6$	No
	$s+1$	$s+1-(w-1)=10>6$	No
	$s+2$	$s+2-(w-1)=11>6$	No
	$s+3$	$s+3-(w-1)=12>6$	No
w	$s-3$	$s-3-(w)=5<6$	Yes
	$s-2$	$s-2-(w)=6\geqslant 6$	Yes
	$s-1$	$s-1-(w)=7>6$	No
	s	$s-(w)=8>6$	No
	$s+1$	$s+1-(w)=9>6$	No
	$s+2$	$s+2-(w)=10>6$	No
	$s+3$	$s+3-(w)=11>6$	No
$w+1$	$s-3$	$s-3-(w+1)=4<6$	Yes
	$s-2$	$s-2-(w+1)=5<6$	Yes
	$s-1$	$s-1-(w+1)=6\geqslant 6$	Yes
	s	$s-(w+1)=7>6$	No
	$s+1$	$s+1-(w+1)=8>6$	No
	$s+2$	$s+2-(w+1)=9>6$	No
	$s+3$	$s+3-(w+1)=10>6$	No
$w+2$	$s-3$	$s-3-(w+2)=3<6$	Yes
	$s-2$	$s-2-(w+2)=4<6$	Yes
	$s-1$	$s-1-(w+2)=5<6$	Yes
	s	$s-(w+2)=6\geqslant 6$	Yes
	$s+1$	$s+1-(w+2)=7>6$	No
	$s+2$	$s+2-(w+2)=8>6$	No
	$s+3$	$s+3-(w+2)=9>6$	No
$w+3$	$s-3$	$s-3-(w+3)=2<6$	Yes
	$s-2$	$s-2-(w+3)=3<6$	Yes
	$s-1$	$s-1-(w+3)=4<6$	Yes
	s	$s-(w+3)=5<6$	Yes
	$s+1$	$s+1-(w+3)=6\geqslant 6$	Yes
	$s+2$	$s+2-(w+3)=7>6$	No
	$s+3$	$s+3-(w+3)=8>6$	No

As a result of the findings in Table 2.9, $n_{ws} = 15$.

Example 2.3.2. In this example, we look at a case when $s - w = 6$ units. In the linear sequence of locations with the user-specified distance equal to 6, when the distance of the centre of clusters C_w and C_s is equal to 6, then there is only one common location random effect. In this example, we take $w = 4$ and $s = 10$. The number of locations belonging to each cluster is $n_4 = n_{10} = 7$.

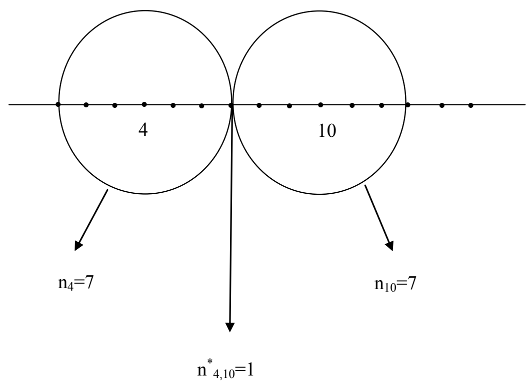

Figure 2.7: Graphical structure of two clusters with one common location random effect.

Table 2.10, shows the number of correlated and uncorrelated uncommon pairs of location random effects. Hence from this table, $n_{4,10} = 15$.

Table 2.10: Correlated uncommon pairs of location random effects.

Locations in C_4 not in C_{10}	Locations in C_{10} not in C_4	Distance of Locations	Correlated Pairs(Yes/No)
1	8	$8-1=7>6$	No
	9	$9-1=8>6$	No
	10	$10-1=9>6$	No
	11	$11-1=10>6$	No
	12	$12-1=11>6$	No
	13	$13-1=12>6$	No
2	8	$8-2=6\geqslant 6$	Yes
	9	$9-2=7>6$	No
	10	$10-2=8>6$	No
	11	$11-2=9>6$	No
	12	$12-2=10>6$	No
	13	$13-2=11>6$	No
3	8	$8-3=5<6$	Yes
	9	$9-3=6\geqslant 6$	Yes
	10	$10-3=7>6$	No
	11	$11-3=8>6$	No
	12	$12-3=9>6$	No
	13	$13-3=10>6$	No
4	8	$8-4=4<6$	Yes
	9	$9-4=5<6$	Yes
	10	$10-4=6\geqslant 6$	Yes
	11	$11-4=7>6$	No
	12	$12-4=8>6$	No
	13	$13-4=9>6$	No
5	8	$8-5=3<6$	Yes
	9	$9-5=4<6$	Yes
	10	$10-5=5<6$	Yes
	11	$11-5=6\geqslant 6$	Yes
	12	$12-5=7>6$	No
	13	$13-5=8>6$	No
6	8	$8-6=2<6$	Yes
	9	$9-6=3<6$	Yes
	10	$10-6=4<6$	Yes
	11	$11-6=5<6$	Yes
	12	$12-6=6\geqslant 6$	Yes
	13	$13-6=7>6$	No

Example 2.3.3. In this example, we consider a case when $s-w = 4$ units. In the linear sequence of locations with the user-specified distance equal to 6, when the distance of the centre of clusters C_w and C_s is equal to 4, then there are three common location random effects. In this example, we take $w = 6$ and $s = 10$. The number of locations belonging to each cluster is $n_6 = n_{10} = 7$.

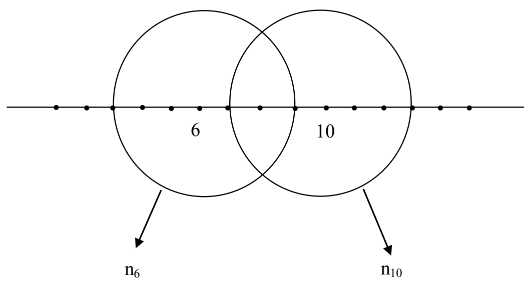

Figure 2.8: Graphical structure of two clusters with three common location random effects.

By using Table 2.11, we can count the correlated uncommon pairs of location random effects. As a result of this table, $n_{6,10} = 6$.

Table 2.11: Correlated uncommon pairs of location random effects.

Locations in C_6 not in C_{10}	Locations in C_{10} not in C_6	Distance of Locations	Correlated Pairs(Yes/No)
3	10	$10-3=7>6$	No
	11	$11-3=8>6$	No
	12	$12-3=9>6$	No
	13	$13-3=10>6$	No
4	10	$10-4=6\geqslant 6$	Yes
	11	$11-4=7>6$	No
	12	$12-4=8>6$	No
	13	$13-4=9>6$	No
5	10	$10-5=5<6$	Yes
	11	$11-5=6\geqslant 6$	Yes
	12	$12-5=7>6$	No
	13	$13-5=8>6$	No
6	10	$10-6=4<6$	Yes
	11	$11-6=5<6$	Yes
	12	$12-6=6\geqslant 6$	Yes
	13	$13-6=7>6$	No

Example 2.3.4. In this example, we consider another case of $s-w=4$ units. Unlike Example 2.3.3, in this case, one of the clusters is centred around the first location in the linear sequence of locations. We assume that $w=1$ and $s=5$. Hence from Figure 2.9, and Tables 2.1 and 2.5 we obtain $n_1=4$, $n_5=7$, and $n^*_{15}=3$.

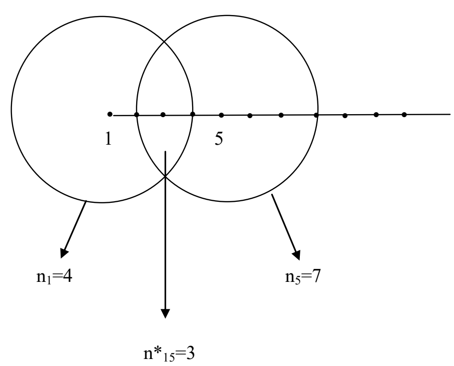

Figure 2.9: Graphical structure of two clusters with three common location random effects.

In Table 2.12, we show that the number of correlated uncommon pairs of location random effects, that is, n_{15}, is 3.

Table 2.12: Correlated uncommon pairs of location random effects.

Locations in C_1 not in C_5	**Locations in C_5 not in C_1**	**Distance of Locations**	**Correlated Pairs(Yes/No)**
1	5	$5-1=4<6$	Yes
	6	$6-1=5<6$	Yes
	7	$7-1=6\geqslant 6$	Yes
	8	$8-1=7>6$	No

2.4 Basic Properties of the Familial-Spatial Linear Mixed Model

From (2.6) and (2.14), the response for the i^{th} member at the s^{th} location can be modeled as,

$$y_{si} = x'_{si}\beta + \frac{1}{\sqrt{n_s}}1'_{n_s}\tilde{\gamma}_s + \alpha_s + \epsilon_{si}, \quad s = 1,2,...,S; \quad i = 1,2,...,m. \tag{2.28}$$

Consider model (2.28) with β as the regression coefficient of x_{si} for all $s = 1,..,S$ and $i = 1,...,m$, and $\tilde{\gamma}_s = (\gamma_{s1},...,\gamma_{sn_s})'$ the vector of location random effects. We relabel the components of the vector of $\tilde{\gamma}_s$ as,

$$\tilde{\gamma}_s = (\gamma_{s1},...,\gamma_{sn_s})' = (\gamma^*_{s-d/2},...,\gamma^*_s,...,\gamma^*_{s+d/2})', \tag{2.29}$$

where $1 \leqslant s - \frac{d}{2} \leqslant S - \frac{d}{2}$ and $1 + \frac{d}{2} \leqslant s + \frac{d}{2} \leqslant S$.

Now, without loss of generality, we assume that $\gamma_s^* = \gamma_{s1}$ such that γ_s^* satisfies a distribution with $E(\gamma_s^*) = 0$, and $var(\gamma_s^*) = \sigma_\gamma^2$ for $s = 1, ..., S$. For any $j, j' \in C_s$ we assume that $\gamma_{sj} = \gamma_j^*$ and $\gamma_{sj'} = \gamma_{j'}^*$, and , we write,

$$corr(\gamma_{sj}, \gamma_{sj'}) = corr(\gamma_j^*, \gamma_{j'}^*) = \delta_{jj'}\phi_{jj'}^* = \phi_{jj'}(s). \tag{2.30}$$

By assuming equi-correlated pair-wise location random effects ($\phi_{jj'}(s) = \phi$), the vector of location random effect $\tilde{\gamma}_s$ has an $n_s \times n_s$ variance-covariance matrix, we denote by Γ_{ss}, defined by

$$var(\tilde{\gamma}_s) = \Gamma_{ss} = \begin{pmatrix} \sigma_\gamma^2 & \sigma_\gamma^2\phi & \dots & \sigma_\gamma^2\phi \\ \sigma_\gamma^2\phi & \sigma_\gamma^2 & \dots & \sigma_\gamma^2\phi \\ \vdots & \vdots & & \vdots \\ \sigma_\gamma^2\phi & \sigma_\gamma^2\phi & \dots & \sigma_\gamma^2 \end{pmatrix}_{n_s \times n_s} = \sigma_\gamma^2 \left(\phi 1_{n_s} 1'_{n_s} + (1-\phi) I_{n_s}\right), \tag{2.31}$$

where $\phi 1_{n_s} 1'_{n_s} + (1-\phi) I_{n_s}$ is the $n_s \times n_s$ correlation matrix of $\tilde{\gamma}_s$ and denoted by $C_{n_s n_s}(\phi)$, therefore,

$$\Gamma_{ss} = \sigma_\gamma^2 C_{n_s n_s}(\phi). \tag{2.32}$$

Based on the decomposition of $\tilde{\gamma}_w$ and $\tilde{\gamma}_s$ in (2.10), the matrix of variance-covariance of $\tilde{\gamma}_w$ and $\tilde{\gamma}_s$ denoted by Γ_{ws}, defined by

$$cov(\tilde{\gamma}_w, \tilde{\gamma}'_s) = \Gamma_{ws} = \begin{pmatrix} cov(\tilde{\gamma}_{w(1)}, \tilde{\gamma}'_{ws}) & cov(\tilde{\gamma}_{w(1)}, \tilde{\gamma}'_{s(2)}) & cov(\tilde{\gamma}_{w(1)}, \tilde{\gamma}'_{s(1)}) \\ cov(\tilde{\gamma}_{w(2)}, \tilde{\gamma}'_{ws}) & cov(\tilde{\gamma}_{w(2)}, \tilde{\gamma}'_{s(2)}) & cov(\tilde{\gamma}_{w(2)}, \tilde{\gamma}'_{s(1)}) \\ cov(\tilde{\gamma}_{ws}, \tilde{\gamma}'_{ws}) & cov(\tilde{\gamma}_{ws}, \tilde{\gamma}'_{s(2)}) & cov(\tilde{\gamma}_{ws}, \tilde{\gamma}'_{s(1)}) \end{pmatrix}_{n_w \times n_s}. \tag{2.33}$$

The matrix of variance-covariance Γ_{ws} can be written as a function of the correlation matrix. Based on the decomposition of the clusters in Section 2.2, Γ_{ws} can be written as,

$$\Gamma_{ws} = \sigma_\gamma^2 \begin{pmatrix} C_{n^*_{w(1)}n^*_{ws}}(\phi) & C_{n^*_{w(1)}n^*_{s(2)}}(\phi) & C_{n^*_{w(1)}n^*_{s(1)}}(\phi) \\ C_{n^*_{w(2)}n^*_{ws}}(\phi) & C_{n^*_{w(2)}n^*_{s(2)}}(\phi) & C_{n^*_{w(2)}n^*_{s(1)}}(\phi) \\ C_{n^*_{ws}n^*_{ws}}(\phi) & C_{n^*_{ws}n^*_{s(2)}}(\phi) & C_{n^*_{ws}n^*_{s(1)}}(\phi) \end{pmatrix}_{n_w \times n_s}. \tag{2.34}$$

From this decomposition of the clusters, none of $n^*_{w(1)}$ random effects are correlated with $n^*_{s(1)}$ and $n^*_{s(2)}$ random effects, and none of $n^*_{s(1)}$ random effects are correlated with $n^*_{w(2)}$ random effects. The challenging component of this matrix that has no closed form matrix is $C_{n^*_{w(2)}n^*_{s(2)}}(\phi)$. Therefore, the components of Γ_{ws} are given by

$$\begin{aligned}
C_{n^*_{w(1)}n^*_{ws}}(\phi) &= corr(\tilde{\gamma}_{w(1)}, \tilde{\gamma}'_{ws}) \ = \phi 1_{n^*_{w(1)}} 1'_{n^*_{ws}} \\
C_{n^*_{w(1)}n^*_{s(2)}}(\phi) &= corr(\tilde{\gamma}_{w(1)}, \tilde{\gamma}'_{s(2)}) = 0_{n^*_{w(1)}n^*_{s(2)}} \\
C_{n^*_{w(1)}n^*_{s(1)}}(\phi) &= corr(\tilde{\gamma}_{w(1)}, \tilde{\gamma}'_{s(1)}) = 0_{n^*_{w(1)}n^*_{s(1)}} \\
C_{n^*_{w(2)}n^*_{ws}}(\phi) &= corr(\tilde{\gamma}_{w(2)}, \tilde{\gamma}'_{ws}) \ = \phi 1_{n^*_{w(2)}} 1'_{n^*_{ws}} \\
C_{n^*_{w(2)}n^*_{s(2)}}(\phi) &= corr(\tilde{\gamma}_{w(2)}, \tilde{\gamma}'_{s(2)}) = \phi L_{n^*_{w(2)}n^*_{s(2)}} \\
C_{n^*_{w(2)}n^*_{s(1)}}(\phi) &= corr(\tilde{\gamma}_{w(2)}, \tilde{\gamma}'_{s(1)}) = 0_{n^*_{w(2)}n^*_{s(1)}} \\
C_{n^*_{ws}n^*_{ws}}(\phi) &= corr(\tilde{\gamma}_{ws}, \tilde{\gamma}'_{ws}) \quad = \phi 1_{n^*_{ws}} 1'_{n^*_{ws}} + (1-\phi) I_{n^*_{ws}} \\
C_{n^*_{ws}n^*_{s(2)}}(\phi) &= corr(\tilde{\gamma}_{ws}, \tilde{\gamma}'_{s(2)}) \ \ = \phi 1_{n^*_{ws}} 1'_{n^*_{s(2)}} \\
C_{n^*_{ws}n^*_{s(1)}}(\phi) &= corr(\tilde{\gamma}_{ws}, \tilde{\gamma}'_{s(1)}) \ \ = \phi 1_{n^*_{ws}} 1'_{n^*_{s(1)}}
\end{aligned} \tag{2.35}$$

where $L_{n^*_{w(2)}n^*_{s(2)}}$ is a $n^*_{w(2)} \times n^*_{s(2)}$ matrix such that some of its entries are one and the rest are zero. As has already been mentioned, not all pairs of random effects in the region $w(2)$ and $s(2)$ are correlated. There are n_{ws} correlated uncommon pairs and $n^*_{w(2)}n^*_{s(2)} - n_{ws}$ uncorrelated uncommon pairs. For the special case of $n^*_{w(2)} = n^*_{s(2)}$, the $L_{n^*_{w(2)}n^*_{s(2)}}$ is a lower triangular matrix with all entries on and below the main diagonal equal to one. Now, for the equi-correlated location random effects, the marginal properties of the vector $\tilde{\gamma}_s$ are

$$E(\tilde{\gamma}_s) = 0_{n_s \times 1}, \tag{2.36}$$

and

$$\tilde{\gamma}_s \sim (0, \Gamma_{ss}), \tag{2.37}$$

where

$$\Gamma_{ss} = \sigma^2_\gamma C_{n_s n_s}(\phi). \tag{2.38}$$

The family random effect, that is, α_s, is identically and independently distributed with $E(\alpha_s) = 0$ and $var(\alpha_s) = \sigma^2_\alpha$ for $s = 1, ..., S$; that is,

$$\alpha_s \overset{iid}{\sim} (0, \alpha^2_\alpha). \tag{2.39}$$

The error model is assumed to follow

$$\epsilon_{si} \overset{iid}{\sim} (0, \sigma^2_\epsilon), \tag{2.40}$$

likewise, ϵ_{si}, α_s, and γ^*_s are assumed to be mutually independent. Now, by using (2.37), (2.39), and (2.40), we can bring these new lemmas for the response variable in (2.28).

Lemma 2.4.1. For the equi-correlated random effect model, the mean and variance of the response variable are given by

$$E(Y_{si}) = x'_{si}\beta = \mu_{si}, \tag{2.41}$$

$$var(Y_{si}) = \sigma^2_\gamma\left[1 + \phi(n_s - 1)\right] + \sigma^2_\alpha + \sigma^2_\epsilon. \tag{2.42}$$

Proof: Because of (2.37), (2.39), and (2.40), it is clear that $E(Y_{si}) = \mu_{si}$. It also follows that

$$\begin{aligned}
var(Y_{si}) =& var(x'_{si}\beta + \frac{1}{\sqrt{n_s}}1'_{n_s}\tilde{\gamma}_s + \alpha_s + \epsilon_{si}) \\
=& var(\frac{1}{\sqrt{n_s}}1'_{n_s}\tilde{\gamma}_s) + var(\alpha_s) + var(\epsilon_{si}) \\
=& \frac{1}{n_s}1'_{n_s}\Gamma_{ss}1_{n_s} + \sigma^2_\alpha + \sigma^2_\epsilon \\
=& \frac{\sigma^2_\gamma}{n_s}1'_{n_s}\Big(\phi 1_{n_s}1'_{n_s} + (1-\phi)I_{n_s}\Big)1_{n_s} + \sigma^2_\alpha + \sigma^2_\epsilon \\
=& \frac{\sigma^2_\gamma}{n_s}\Big(\phi n_s^2 + (1-\phi)n_s\Big) + \sigma^2_\alpha + \sigma^2_\epsilon \\
=& \sigma^2_\gamma\left[1 + \phi(n_s - 1)\right] + \sigma^2_\alpha + \sigma^2_\epsilon
\end{aligned} \tag{2.43}$$

or

$$\begin{aligned}
var(Y_{si}) =& \frac{1}{n_s}\left[\sum_{j=1}^{n_s} var(\gamma_{sj}) + 2\sum_{j'<j}^{n_s} cov(\gamma_{sj'}, \gamma_{sj})\right] + \sigma^2_\alpha + \sigma^2_\epsilon \\
=& \frac{1}{n_s}\left[n_s\sigma^2_\gamma + 2\binom{n_s}{2}\sigma^2_\gamma\phi\right] + \sigma^2_\alpha + \sigma^2_\epsilon \\
=& \sigma^2_\gamma\left[1 + \phi(n_s - 1)\right] + \sigma^2_\alpha + \sigma^2_\epsilon.
\end{aligned} \tag{2.44}$$

If the location random effects are independent, it then follows that

$$var(Y_{si}) = \sigma^2_\gamma + \sigma^2_\alpha + \sigma^2_\epsilon. \tag{2.45}$$

■

Since at any spatial location there are m responses, the covariance of responses consists of two parts. The first part is the covariances of different responses at the same location, that is, $cov(Y_{si}, Y_{si'})$ for $s = 1, 2, ..., S$; $(i \neq i')$, and the second part is the covariances of responses from two different locations, that is, $cov(Y_{wi}, Y_{si'})$ for $w \neq s$.

Lemma 2.4.2. The covariance of two responses from the same location is given by

$$cov(Y_{si}, Y_{si'}) = \sigma_\gamma^2 \left[1 + \phi(n_s - 1)\right] + \sigma_\alpha^2. \tag{2.46}$$

Proof: Because $x'_{si}\beta$ is a deterministic component and ϵ_{si} is identically and independently distributed, it then follows that

$$\begin{aligned} cov(Y_{si}, Y_{si'}) &= cov(\frac{1}{\sqrt{n_s}} 1'_{n_s}\tilde{\gamma}_s + \alpha_s + \epsilon_{si} \ , \ \ \frac{1}{\sqrt{n_s}} 1'_{n_s}\tilde{\gamma}_s + \alpha_s + \epsilon_{si'}) \\ &= \frac{1}{n_s} 1'_{n_s}\Gamma_{ss} 1_{n_s} + \sigma_\alpha^2 \\ &= \sigma_\gamma^2 \left[1 + \phi(n_s - 1)\right] + \sigma_\alpha^2 \end{aligned}$$

or

$$\begin{aligned} cov(Y_{si}, Y_{si'}) &= \frac{1}{n_s} \left[\sum_{j=1}^{n_s} var(\gamma_{sj}) + 2 \sum_{j'<j}^{n_s} cov(\gamma_{sj'}, \gamma_{sj}) \right] + \sigma_\alpha^2 \\ &= \frac{1}{n_s} \left[n_s \sigma_\gamma^2 + 2 \binom{n_s}{2} \sigma_\gamma^2 \phi \right] + \sigma_\alpha^2 \\ &= \sigma_\gamma^2 \left[1 + \phi(n_s - 1)\right] + \sigma_\alpha^2. \end{aligned}$$

Note that if the location random effects are independent, it then follows that

$$cov(Y_{si}, Y_{si'}) = \sigma_\gamma^2 + \sigma_\alpha^2. \tag{2.47}$$

■

Lemma 2.4.3. The covariance of the two responses from different locations ($w \neq s$) is given by

$$cov(Y_{wi}, Y_{si'}) = \frac{\sigma_\gamma^2}{\sqrt{n_w n_s}} \Big\{ n_{ws}^* + \phi \Big(n_{ws} + n_{ws}^*(n_w^* + n_s^*) + n_{ws}^*(n_{ws}^* - 1) \Big) \Big\}. \tag{2.48}$$

Proof: Because of (2.39) and (2.40), we can write

$$\begin{aligned} cov(Y_{wi}, Y_{si'}) =& cov(\frac{1}{\sqrt{n_w}} 1'_{n_w} \tilde{\gamma}_w + \alpha_w + \epsilon_{wi} \quad , \quad \frac{1}{\sqrt{n_s}} 1'_{n_s} \tilde{\gamma}_s + \alpha_s + \epsilon_{si'}) \\ =& \frac{1}{\sqrt{n_w n_s}} 1'_{n_w} \Gamma_{ws} 1_{n_s}. \end{aligned} \tag{2.49}$$

From the correlation structure of location random effects of two distinct clusters and components of the correlation matrix in (2.35), we can write

$$\begin{aligned} cov(Y_{wi}, Y_{si'}) =& \frac{\sigma_\gamma^2}{\sqrt{n_w n_s}} \Big\{ \phi 1'_{n^*_{w(1)}} 1_{n^*_{w(1)}} 1'_{n^*_{ws}} 1_{n^*_{ws}} + \phi 1'_{n^*_{w(2)}} 1_{n^*_{w(2)}} 1'_{n^*_{ws}} 1_{n^*_{ws}} \\ & + 1'_{n^*_{ws}} \Big(\phi 1_{n^*_{ws}} 1'_{n^*_{ws}} + (1-\phi) I_{n^*_{ws}} \Big) 1_{n^*_{ws}} + 1'_{n^*_{w(1)}} C_{n^*_{w(1)} n^*_{s(2)}} 1_{n^*_{s(2)}} \\ & + \phi 1'_{n^*_{w(2)}} L_{n^*_{w(2)} n^*_{s(2)}} 1_{n^*_{s(2)}} + \phi 1'_{n^*_{ws}} 1_{n^*_{ws}} 1'_{n^*_{s(2)}} 1_{n^*_{s(2)}} \\ & + 1'_{n^*_{w(1)}} C_{n^*_{w(1)} n^*_{s(1)}} 1_{n^*_{s(1)}} + 1'_{n^*_{w(2)}} C_{n^*_{w(2)} n^*_{s(1)}} 1_{n^*_{s(1)}} \\ & + \phi 1'_{n^*_{ws}} 1_{n^*_{ws}} 1'_{n^*_{s(1)}} 1_{n^*_{s(1)}} \Big\}. \end{aligned} \tag{2.50}$$

The number of location random effects in each of the classifications is identified in

Section 2.2. Note that $L_{n^*_{w(2)}n^*_{s(2)}}$ is a $n^*_{w(2)} \times n^*_{s(2)}$ matrix with the number of n_{ws} entries of one, therefore we have

$$\phi 1'_{n^*_{w(2)}} L_{n^*_{w(2)}n^*_{s(2)}} 1_{n^*_{s(2)}} = \phi n_{ws}. \tag{2.51}$$

It then follows that

$$\begin{aligned}
cov(Y_{wi}, Y_{si'}) =& \frac{\sigma^2_\gamma}{\sqrt{n_w n_s}} \Big\{ \phi n^*_{w(1)} n^*_{ws} + \phi n^*_{w(2)} n^*_{ws} + \phi n^*_{ws} n^*_{ws} + (1-\phi) n^*_{ws} \\
&+ \phi n_{ws} + \phi n^*_{ws} n^*_{s(2)} + \phi n^*_{ws} n^*_{s(1)} \Big\} \\
=& \frac{\sigma^2_\gamma}{\sqrt{n_w n_s}} \Big\{ \phi n^*_{ws} (n^*_{w(1)} + n^*_{w(2)}) + \phi n^*_{ws} (n^*_{s(1)} + n^*_{s(2)}) \\
&+ \phi n_{ws} + n^*_{ws} + \phi n^*_{ws} (n^*_{ws} - 1) \Big\} \\
=& \frac{\sigma^2_\gamma}{\sqrt{n_w n_s}} \Big\{ \phi n^*_{ws} (n^*_w + n^*_s) + \phi n_{ws} + n^*_{ws} + \phi n^*_{ws} (n^*_{ws} - 1) \Big\} \\
=& \frac{\sigma^2_\gamma}{\sqrt{n_w n_s}} \Big\{ n^*_{ws} + \phi \Big(n_{ws} + n^*_{ws} (n^*_w + n^*_s) + n^*_{ws} (n^*_{ws} - 1) \Big) \Big\}.
\end{aligned} \tag{2.52}$$

Note that for the independent location random effects, the covariance in (2.52) is equivalent to

$$cov(Y_{wi}, Y_{si'}) = \sigma^2_\gamma \frac{n^*_{ws}}{\sqrt{n_w n_s}}. \tag{2.53}$$

■

Now, from Lemmas (2.4.1), (2.4.2), and (2.4.3), we use Σ to denote the matrix of variance-covariance of responses, so that

$$cov(Y_{wi}, Y_{si'}) = \Sigma = \begin{cases} \sigma_\gamma^2\left[1+\phi(n_s-1)\right]+\sigma_\alpha^2+\sigma_\epsilon^2, & w=s, i=i' \\ \sigma_\gamma^2\left[1+\phi(n_s-1)\right]+\sigma_\alpha^2, & w=s, i\neq i' \\ \frac{\sigma_\gamma^2}{\sqrt{n_w n_s}}\Big[n_{ws}^* & w\neq s, i\neq i' \\ \quad +\phi(n_{ws}+n_{ws}^*(n_w^*+n_s^*)+n_{ws}^*(n_{ws}^*-1)\Big]. \end{cases} \tag{2.54}$$

For the special case of independent location random effects, the matrix of variance-covariance reduces to

$$cov(Y_{wi}, Y_{si'}) = \Sigma = \begin{cases} \sigma_\gamma^2+\sigma_\alpha^2+\sigma_\epsilon^2, & w=s, i=i' \\ \sigma_\gamma^2+\sigma_\alpha^2, & w=s, i\neq i' \\ \frac{n_{ws}^*}{\sqrt{n_w n_s}}\sigma_\gamma^2, & w\neq s, i\neq i'. \end{cases} \tag{2.55}$$

In the next chapter, we begin to consider the estimation of parameters of model (2.28) when location random effects are equi-correlated. Then, we provide the estimation for model parameters when location random effects are independent.

Chapter 3

Estimation in Cluster-Based Familial-Spatial Linear Mixed Effect Model

In Chapter 2, we derived the marginal properties of the response variable in Section 2.4. For convenience, we assumed that at each location there was an equal number of observations. We mentioned that the estimation of the spatial linear mixed effect model parameters with one observation at each location has been discussed by Mariathas and Sutradhar (2016). This chapter involves the estimation of the unknown parameters of the proposed model (2.28), including regression effect β, the variance of location random effects σ^2_γ, the variance of family random effects σ^2_α, correlation of location random effects ϕ, and variance of error model σ^2_ϵ. In order to carry out the parameter estimations, it is helpful to write the model (2.28) in matrix notation.

3.1 Model and Estimation

Consider the linear mixed effect model

$$Y = X\beta + \Omega \underset{\sim}{R} + \alpha + \epsilon, \tag{3.1}$$

where $Y = (y_{11}, ..., y_{1m}, ..., y_{S1}, ..., y_{Sm})'$ is the $mS \times 1$ vector of response variables, $X = (x'_{11}, ..., x'_{1m}, ..., x'_{S1}, ..., x'_{Sm})'$ is the $mS \times p$ design matrix with $x_{si} = (x_{si1}, ..., x_{sip})'$ for $s = 1, 2, ..., S$ and $i = 1, 2, ..., m$. Let $\beta \in \mathbb{R}^p$ be the $p \times 1$ vector of unknown regression effects. In addition, Ω is the weight matrix such that each row of a block diagonal matrix is the weight vector $\frac{1}{\sqrt{n_s}} 1'_{n_s}$ corresponding to a specific location for $s = 1, 2, ..., S$. By assuming that $\sum_{s=1}^{S} n_s = N$, we have

$$\Omega = \begin{pmatrix} \begin{matrix} \frac{1}{\sqrt{n_1}} & \cdots & \frac{1}{\sqrt{n_1}} \\ \vdots & & \vdots \\ \frac{1}{\sqrt{n_1}} & \cdots & \frac{1}{\sqrt{n_1}} \end{matrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{matrix} \frac{1}{\sqrt{n_S}} & \cdots & \frac{1}{\sqrt{n_S}} \\ \vdots & & \vdots \\ \frac{1}{\sqrt{n_S}} & \cdots & \frac{1}{\sqrt{n_S}} \end{matrix} \end{pmatrix}_{mS \times N}.$$

In (3.1), $\underset{\sim}{R}$ is an $N \times 1$ vector of all S location random effect vectors, that is, $\underset{\sim}{R} = (\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_S)'$. Also, $\alpha = (\alpha_1, ..., \alpha_1, \alpha_2, ..., \alpha_2, ..., \alpha_S, ..., \alpha_S)'$ is a $mS \times 1$ vector of all family random effect with $E(\alpha) = 0$, and $mS \times mS$ variance-covariance matrix such that all entries of block diagonal sub-matrices are σ_α^2 and the rest of the entries are zero. That is,

$$var(\alpha) = \left(\begin{array}{c|c|c} \sigma_\alpha^2(11')_{m\times m} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \sigma_\alpha^2(11')_{m\times m} \end{array}\right)_{mS\times mS}.$$

Further, $\epsilon = (\epsilon_{11}, ..., \epsilon_{1m}, ..., \epsilon_{S1}, ..., \epsilon_{Sm})'$ is the $mS \times 1$ vector of error terms with $E(\epsilon) = 0$, and $cov(\epsilon) = \sigma_\epsilon^2 I_{mS}$. Under the above assumptions and Lemma 2.4.1, we have

$$\mu(\beta) = E(Y) = X\beta = (\mu_{11}(\beta), ..., \mu_{1m}(\beta), ..., \mu_{si}(\beta), ..., \mu_{Sm}(\beta))', \tag{3.2}$$

where $\mu_{si}(\beta) = E(Y_{si}) = x'_{si}\beta$, and $\Sigma = cov(Y) = f(\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2, \phi)$. Recall that the entries of Σ were obtained by (2.54). Clearly, $\mu(\beta)$ is a function of unknown β, and Σ is a function of the unknown scale and correlation parameters. In our proposed model (2.28), we are dealing with the estimation of $p + 4$ parameters simultaneously, which includes p parameters of regression effects $\beta = (\beta_1, ..., \beta_p)'$, three scale parameters σ_γ^2, σ_α^2, σ_ϵ^2, and one correlation parameter ϕ. In one special case, if we assume that the location random effects are independent ($\phi = 0$), then the number of parameters reduces to $p + 3$.

3.2 Generalized Least Squares (GLS) Estimation

3.2.1 Estimation of Regression Parameter β

In this section, we develop a GLS estimation method for β under a familial-spatial random effect model. Since the familial-spatial responses are correlated, to have a more efficient estimator for β, the so-called GLS method should be applied. Although both OLS and GLS estimators are unbiased for β, the GLS estimator has a smaller variance than OLS (Amemiya 1985, Section 6.1.3). In this approach, to obtain a GLS estimate of β, the following generalized sum of the square should be minimized; that is,

$$\begin{aligned} L(\beta) &= (Y - X\beta)'\Sigma^{-1}(Y - X\beta) \\ &= Y'\Sigma^{-1}Y - 2\beta'X'\Sigma^{-1}Y + \beta'X'\Sigma^{-1}X\beta. \end{aligned} \tag{3.3}$$

Let the GLS estimator of β be denoted by $\hat{\beta}_{gls}$. It is clear that $\hat{\beta}_{gls}$ can be obtained by solving

$$\frac{\partial L(\beta)}{\partial \beta} = 0. \tag{3.4}$$

Thus we have

$$\hat{\beta}_{gls} = \left[X'\Sigma^{-1}X\right]^{-1}\left[X'\Sigma^{-1}Y\right], \tag{3.5}$$

where by (3.2), one obtains $E(\hat{\beta}_{gls}) = \beta$. This means that $\hat{\beta}_{gls}$ in (3.5) is an unbiased estimator of β if the components of Σ are known. Moreover, $var(\hat{\beta}_{gls}) = (X'\Sigma^{-1}X)^{-1}$ is a function of the unknown parameters σ^2_γ, σ^2_α, σ^2_ϵ, and ϕ. Therefore, to estimate β, the components of Σ need to be estimated first. The following theorem states that the GLS estimate of β is more efficient than the OLS estimates.

Theorem 3.2.1. Let $\hat{\beta}_{ols} = (\hat{\beta}_{1,ols}, ..., \hat{\beta}_{p,ols})'$ and $\hat{\beta}_{gls} = (\hat{\beta}_{1,gls}, ..., \hat{\beta}_{p,gls})'$ denote the OLS and the GLS estimators of vector $\beta = (\beta_1, ..., \beta_p)'$, respectively. Then for each $i = 1, ..., p$, it follows that:

$$cov(\hat{\beta}_{i,gls}) \leqslant cov(\hat{\beta}_{i,ols}).$$

Proof: Let $A = (X'X)^{-1}$ and $B = (X'\Sigma^{-1}X)^{-1}$. Then,

$$\begin{aligned}
cov(\hat{\beta}_{ols}) &= cov\left[(X'X)^{-1}X'Y\right] \\
&= cov\left[(X'X)^{-1}X'Y - (X'\Sigma^{-1}X)^{-1}(X'\Sigma^{-1}Y) + (X'\Sigma^{-1}X)^{-1}(X'\Sigma^{-1}Y)\right] \\
&= cov\left[AX'Y - B(X'\Sigma^{-1}Y)\right] + cov(\hat{\beta}_{gls}) \\
&+ 2cov\Big\{\left[AX'Y - B(X'\Sigma^{-1}Y)\right], B(X'\Sigma^{-1}Y)\Big\}.
\end{aligned}$$

It is clear that,

$$cov\Big\{\left[AX'Y - B(X'\Sigma^{-1}Y)\right], B(X'\Sigma^{-1}Y)\Big\} \tag{3.6}$$

$$\begin{aligned}
&= cov\left[AX'Y, B(X'\Sigma^{-1}Y)\right] - var\left[B(X'\Sigma^{-1}Y)\right] \\
&= AX'\Sigma\Sigma^{-1}XB' - BX'\Sigma^{-1}XB' \\
&= B' - B' = 0.
\end{aligned}$$

Therefore for each element of vector β we have $cov(\hat{\beta}_{i,gls}) \leqslant cov(\hat{\beta}_{i,ols})$.

■

In the next section, we apply the moment (MM) method to estimate the scale and correlation parameters. We will consider two scenarios:

- (i) when the location random effects are independent ($\phi^*_{ws} = 0$), and
- (ii) when the location random effects are equi-correlated ($\phi^*_{ws} = \phi$).

3.2.2 Method of Moment Approach for $\xi = (\sigma^2_\gamma, \sigma^2_\alpha, \sigma^2_\epsilon, \phi)'$

3.2.2.1 Equi-Correlated Location Random Effects $\phi^*_{ws} = \phi$

For the equi-correlated location random effects model, from (2.54), the components of Σ are the functions of unknown scale and correlation parameters. Correlation between two location random effects belonging to the same cluster is given by

$$corr(\gamma_{si}, \gamma_{si'}) = \begin{cases} 1, & \text{for } i = i' \\ \phi, & \text{for } i \neq i' \end{cases} \tag{3.7}$$

where γ_{si} is the i^{th} component of the vector of location random effect $\tilde{\gamma}_s$. To estimate the model's scale and correlation parameters, it is necessary to solve four unbiased moment estimating equations simultaneously. Generally speaking, one of the common challenges of the moment estimation approach with multiple parameters is detecting the appropriate unbiased statistics for the model's parameters. In addition, as the distance between two locations increases in the spatial setup, the correlation of responses measured at these locations decreases. So, lag zero correlation (sample variance), lag one correlation, lag two correlation, and sample covariance are the suggested estimating equations that are given below as,

$$W_1 = \sum_{s=1}^{S}\sum_{i=1}^{m}(y_{si}-\mu_{si})^2/mS; \quad \text{lag 0 correlation} \tag{3.8a}$$

$$W_2 = \frac{\sum\limits_{s=1}^{S-1}\sum\limits_{i=1}^{m}\sum\limits_{i'=1}^{m}(y_{si}-\mu_{si})(y_{(s+1)i'}-\mu_{(s+1)i'})/m^2(S-1)}{\sum\limits_{s=1}^{S}\sum\limits_{i=1}^{m}(y_{si}-\mu_{si})^2/mS}; \quad \text{lag one correlation} \tag{3.8b}$$

$$W_3 = \frac{\sum\limits_{s=1}^{S-2}\sum\limits_{i=1}^{m}\sum\limits_{i'=1}^{m}(y_{si}-\mu_{si})(y_{(s+2)i'}-\mu_{(s+2)i'})/m^2(S-2)}{\sum\limits_{s=1}^{S}\sum\limits_{i=1}^{m}(y_{si}-\mu_{si})^2/mS}; \quad \text{lag two correlation} \tag{3.8c}$$

$$W_4 = \frac{1}{\binom{m}{2}S}\sum_{s=1}^{S}\sum_{i=1}^{m-1}\sum_{i'>i}^{m}(y_{si}-\mu_{si})(y_{si'}-\mu_{si'}); \quad \text{sample covariance.} \tag{3.8d}$$

The 4-dimensional vector of these statistics is given by $W = (W_1, W_2, W_3, W_4)'$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)' = E(W)$. To obtain the moment estimates of $\xi = (\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2, \phi)'$, we need to solve the following unbiased estimating equation

$$W - \lambda = 0. \tag{3.9}$$

The equation (3.9) contains four sub-equations of $W_i - \lambda_i = 0$ for $i = 1, 2, 3, 4$. We obtain the expectation of W_i by using (2.54) as follows:

$$\begin{aligned}\lambda_1 = E(W_1) =& E\Big(\frac{1}{mS}\sum_{s=1}^{S}\sum_{i=1}^{m}(y_{si}-\mu_{si})^2\Big)\\ =&\frac{1}{mS}\sum_{s=1}^{S}\sum_{i=1}^{m}var(Y_{si})\\ =&\frac{1}{mS}\sum_{s=1}^{S}\sum_{i=1}^{m}\big[\sigma_\gamma^2(1+(n_s-1)\phi)+\sigma_\alpha^2+\sigma_\epsilon^2\big]\\ =&\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\epsilon^2+\frac{\sigma_\gamma^2\phi}{S}\sum_{s=1}^{S}(n_s-1)\\ =&\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\epsilon^2+\frac{\sigma_\gamma^2\phi}{S}(N-S). \end{aligned} \tag{3.10}$$

The expected values of W_2 and W_3 are approximately

$$\begin{aligned}\lambda_2 = E(W_2) \cong& \frac{E\Big(\frac{1}{m^2(S-1)}\sum_{s=1}^{S-1}\sum_{i=1}^{m}\sum_{i'=1}^{m}(y_{si}-\mu_{si})(y_{(s+1)i'}-\mu_{(s+1)i'})\Big)}{E(W_1)}\\ =& \frac{1}{m^2(S-1)\lambda_1}\sum_{s=1}^{S-1}\frac{m^2\sigma_\gamma^2}{\sqrt{n_s n_{s+1}}}\Big(n^*_{s,s+1}\\ &+\phi\big(n_{s,s+1}+n^*_{s,s+1}(n^*_s+n^*_{s+1})+n^*_{s,s+1}(n^*_{s,s+1}-1)\big)\Big)\\ =&\frac{\sigma_\gamma^2}{\lambda_1}\lambda_{21} \end{aligned} \tag{3.11}$$

and,

$$\lambda_3 = E(W_3) \cong \frac{E\Big(\frac{1}{m^2(S-2)} \sum_{s=1}^{S-2} \sum_{i=1}^{m} \sum_{i'=1}^{m} (y_{si} - \mu_{si})(y_{(s+2)i'} - \mu_{(s+2)i'})\Big)}{E(W_1)}$$

$$= \frac{1}{m^2(S-2)\lambda_1} \sum_{s=1}^{S-2} \frac{m^2\sigma_\gamma^2}{\sqrt{n_s n_{s+2}}} \Big(n^*_{s,s+2}$$

$$+ \phi\big(n_{s,s+2} + n^*_{s,s+2}(n^*_s + n^*_{s+2}) + n^*_{s,s+2}(n^*_{s,s+2} - 1)\big)\Big)$$

$$= \frac{\sigma_\gamma^2}{\lambda_1}\lambda_{31} \tag{3.12}$$

where

$$\lambda_{21} = \frac{1}{(S-1)} \sum_{s=1}^{S-1} \frac{1}{\sqrt{n_s n_{s+1}}} \Big(n^*_{s,s+1}$$

$$+ \phi\big(n_{s,s+1} + n^*_{s,s+1}(n^*_s + n^*_{s+1}) + n^*_{s,s+1}(n^*_{s,s+1} - 1)\big)\Big), \tag{3.13}$$

and,

$$\lambda_{31} = \frac{1}{(S-2)} \sum_{s=1}^{S-2} \frac{1}{\sqrt{n_s n_{s+2}}} \Big(n^*_{s,s+2}$$

$$+ \phi\big(n_{s,s+2} + n^*_{s,s+2}(n^*_s + n^*_{s+2}) + n^*_{s,s+2}(n^*_{s,s+2} - 1)\big)\Big).$$

(3.14)

Also, we have

$$\lambda_4 = E(W_4) = E\Big(\frac{1}{\binom{m}{2} S} \sum_{s=1}^{S} \sum_{i=1}^{m-1} \sum_{i'>i}^{m} (y_{si} - \mu_{si})(y_{si'} - \mu_{si'})\Big)$$

$$= \frac{1}{S} \sum_{s=1}^{S} \big[\sigma^2_\gamma + \sigma^2_\alpha + \sigma^2_\gamma (n_s - 1)\phi\big]$$

$$= \sigma^2_\gamma + \sigma^2_\alpha + \frac{\sigma^2_\gamma \phi (N - S)}{S}. \tag{3.15}$$

The moment estimates of components of ξ can be obtained by solving the following so-called Newton-Raphson iterative equation

$$\hat{\xi}_{MM}(r+1) = \hat{\xi}_{MM}(r) + P^{-1}_{(r)} (W - \lambda)_{(r)}, \tag{3.16}$$

where $(.)_r$ denotes the value of expression at the r^{th} iteration, and $P_{(r)}$ is the 4×4 first derivative matrix of λ with respect to the components of ξ. Let $\hat{\xi}_{MM}$ be the final solution of (3.16). Hence, we have

$$P = \begin{bmatrix} \partial\lambda_1/\partial\sigma_\gamma^2 & \partial\lambda_1/\partial\sigma_\alpha^2 & \partial\lambda_1/\partial\sigma_\epsilon^2 & \partial\lambda_1/\partial\phi, \\ \partial\lambda_2/\partial\sigma_\gamma^2 & \partial\lambda_2/\partial\sigma_\alpha^2 & \partial\lambda_2/\partial\sigma_\epsilon^2 & \partial\lambda_2/\partial\phi, \\ \partial\lambda_3/\partial\sigma_\gamma^2 & \partial\lambda_3/\partial\sigma_\alpha^2 & \partial\lambda_3/\partial\sigma_\epsilon^2 & \partial\lambda_3/\partial\phi, \\ \partial\lambda_4/\partial\sigma_\gamma^2 & \partial\lambda_4/\partial\sigma_\alpha^2 & \partial\lambda_4/\partial\sigma_\epsilon^2 & \partial\lambda_4/\partial\phi, \end{bmatrix}. \tag{3.17}$$

The elements of the derivative matrix P can be obtained by using equations (3.10), (3.11), (3.12), and (3.15) for λ_1, λ_2, λ_3, and λ_4, respectively. We provide the derivative equations of matrix P as follows. The derivatives with respect to σ_γ^2 are:

$$\begin{aligned} \frac{\partial\lambda_1}{\partial\sigma_\gamma^2} &= 1 + \frac{\phi(N-S)}{S}, \\ \frac{\partial\lambda_2}{\partial\sigma_\gamma^2} &= \frac{(\sigma_\alpha^2 + \sigma_\epsilon^2)\lambda_{21}}{\lambda_1^2}, \\ \frac{\partial\lambda_3}{\partial\sigma_\gamma^2} &= \frac{(\sigma_\alpha^2 + \sigma_\epsilon^2)\lambda_{31}}{\lambda_1^2}, \\ \frac{\partial\lambda_4}{\partial\sigma_\gamma^2} &= 1 + \frac{\phi(N-S)}{S}. \end{aligned}$$

The derivatives with respect to σ_α^2 are:

$$\begin{aligned} \frac{\partial\lambda_1}{\partial\sigma_\alpha^2} &= 1, \\ \frac{\partial\lambda_2}{\partial\sigma_\alpha^2} &= \frac{-\sigma_\gamma^2\lambda_{21}}{\lambda_1^2}, \\ \frac{\partial\lambda_3}{\partial\sigma_\alpha^2} &= \frac{-\sigma_\gamma^2\lambda_{31}}{\lambda_1^2}, \\ \frac{\partial\lambda_4}{\partial\sigma_\alpha^2} &= 1. \end{aligned}$$

The derivatives with respect to σ_ϵ^2 are:

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial \sigma_\epsilon^2} &= 1, \\
\frac{\partial \lambda_2}{\partial \sigma_\epsilon^2} &= \frac{-\sigma_\gamma^2 \lambda_{21}}{\lambda_1^2}, \\
\frac{\partial \lambda_3}{\partial \sigma_\epsilon^2} &= \frac{-\sigma_\gamma^2 \lambda_{31}}{\lambda_1^2}, \\
\frac{\partial \lambda_4}{\partial \sigma_\epsilon^2} &= 0.
\end{aligned}$$

Finally, the derivatives with respect to ϕ are:

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial \phi} &= \frac{\sigma_\gamma^2 (N-S)}{S}, \\
\frac{\partial \lambda_2}{\partial \phi} &= \frac{\sigma_\gamma^2}{\lambda_1}\lambda_{22} - \frac{\sigma_\gamma^4 \lambda_{21}}{S\lambda_1^2}(N-S), \\
\frac{\partial \lambda_3}{\partial \phi} &= \frac{\sigma_\gamma^2}{\lambda_1}\lambda_{33} - \frac{\sigma_\gamma^4 \lambda_{31}}{S\lambda_1^2}(N-S), \\
\frac{\partial \lambda_4}{\partial \phi} &= \frac{\sigma_\gamma^2 (N-S)}{S},
\end{aligned}$$

where λ_{21} and λ_{31} are given in (3.13) and (3.14), respectively. Further, we have

$$\lambda_{22} = \frac{\partial \lambda_{21}}{\partial \phi} = \frac{1}{(S-1)} \sum_{s=1}^{S-1} \frac{1}{\sqrt{n_s n_{s+1}}} \left[n_{s,s+1} + n_{s,s+1}^*(n_s^* + n_{s+1}^*) + n_{s,s+1}^*(n_{s,s+1}^* - 1)) \right],$$

and

$$\lambda_{33} = \frac{\partial \lambda_{31}}{\partial \phi} = \frac{1}{(S-2)} \sum_{s=1}^{S-2} \frac{1}{\sqrt{n_s n_{s+2}}} \left[n_{s,s+2} + n_{s,s+2}^*(n_s^* + n_{s+2}^*) + n_{s,s+2}^*(n_{s,s+2}^* - 1)) \right].$$

3.2.2.2 Computational Steps

Note that the GLS estimate of β can be obtained from (3.5), and MM estimates of σ_γ^2, σ_α^2,σ_ϵ^2, and ϕ can be obtained from estimating equation (3.16). For the purpose of parameter estimations, we are exploiting these steps.

Step 1: For the appropriate set of initial values for the scale and correlation parameters $\xi^{(0)} = (\sigma_\gamma^{2\,(0)}, \sigma_\alpha^{2\,(0)}, \sigma_\epsilon^{2\,(0)}, \phi^{(0)})'$, we construct Σ from (2.54) and then estimate β (say $\beta^{(0)}$) by using (3.5).

Step 2: With gained $\beta^{(0)}$ in Step 1, we calculate σ_γ^2, σ_α^2, σ_ϵ^2, and ϕ through the estimating equation (3.16). This new set of scale and correlation parameters becomes $\xi^{(1)} = (\sigma_\gamma^{2\,(1)}, \sigma_\alpha^{2\,(1)}, \sigma_\epsilon^{2\,(1)}, \phi^{(1)})'$.

Step 3: Now use these new values of scale and correlation parameters from Step 2 in Step 1 to update β (say $\beta^{(1)}$). Then $\beta^{(1)}$ is used in Step 2 to improve ξ. This iteration cycle continues until convergence.

3.2.2.3 Independent Location Random Effects $\phi_{ws}^* = \phi = 0$

We consider this situation because when two locations are far apart, the random effects of these locations are uncorrelated. It implies that the responses from each location are only affected by the random effect of its location. For the independent location random effects and from equation (2.55), the variance-covariance component of responses are given by

$$cov(Y_{wi'}, Y_{si}) = \Sigma = \begin{cases} \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\epsilon^2 = \sigma^2, & w = s, i' = i \\ \sigma_\gamma^2 + \sigma_\alpha^2 = \sigma_0^2, & w = s, i' \neq i \\ \frac{n_{ws}^*}{\sqrt{n_w n_s}} \sigma_\gamma^2 = \frac{n_{ws}^*}{\sqrt{n_w n_s}} \tau \sigma^2, & w \neq s, i' \neq i \end{cases} \tag{3.18}$$

where $\tau = \frac{\sigma_\gamma^2}{\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\epsilon^2} = \frac{\sigma_\gamma^2}{\sigma^2}$. One of the main reasons for using τ is that τ represents the proportion of variation in the responses, which is explained by the location random effect. Further, it is clear from (3.18) that the true pair-wise familial-spatial correlation of responses is given by

$$\rho_{w,s} = corr(Y_{wi'}, Y_{si}) = \frac{\frac{n_{ws}^*}{\sqrt{n_w n_s}}\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\epsilon^2} = \frac{n_{ws}^*}{\sqrt{n_w n_s}}\tau, \tag{3.19}$$

therefore, τ is an important parameter for estimating $\rho_{w,s}$.

In order to develop the moment estimate of the variance of responses, which is σ^2, we use sample variance. To estimate the covariance of responses at the same location, which is σ_0^2, we apply sample covariance at the same location. To estimate τ, we use sample lag one correlation or sample lag two correlation. Therefore, three estimating equations need to be solved simultaneously. The solutions are provided in the following lemma.

Lemma 3.2.2. An approximated unbiased estimators of σ^2, σ_0^2, τ (using lag one correlation), and τ (using lag two correlation) are given by

$$\hat{\sigma}^2 = \frac{1}{mS}\sum_{s=1}^{S}\sum_{i=1}^{m}(y_{si} - \mu_{si})^2, \tag{3.20}$$

$$\hat{\sigma}_0^2 = \frac{1}{\binom{m}{2}S}\sum_{s=1}^{S}\sum_{i=1}^{m-1}\sum_{i'>i}^{m}(y_{si} - \mu_{si})(y_{si'} - \mu_{si'}), \tag{3.21}$$

$$\hat{\tau} = \frac{W_2(S-1)}{\sum_{s=1}^{S-1} \frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}}, \qquad \text{by using lag one correlation} \tag{3.22a}$$

$$\hat{\tau} = \frac{W_3(S-2)}{\sum_{s=1}^{S-2} \frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}}, \qquad \text{by using lag two correlation} \tag{3.22b}$$

respectively. Recall from (3.8b) and (3.8c) that

$$W_2 = \frac{\sum_{s=1}^{S-1} \sum_{i=1}^{m} \sum_{i'=1}^{m} (y_{si} - \mu_{si})(y_{(s+1)i'} - \mu_{(s+1)i'})/m^2(S-1)}{\sum_{s=1}^{S} \sum_{i=1}^{m} (y_{si} - \mu_{si})^2/mS},$$

and

$$W_3 = \frac{\sum_{s=1}^{S-2} \sum_{i=1}^{m} \sum_{i'=1}^{m} (y_{si} - \mu_{si})(y_{(s+2)i'} - \mu_{(s+2)i'})/m^2(S-2)}{\sum_{s=1}^{S} \sum_{i=1}^{m} (y_{si} - \mu_{si})^2/mS}.$$

Proof: Clearly, as $E(Y_{si}) = \mu_{si}$, then

$$\begin{aligned} E(\hat{\sigma}^2) &= \frac{1}{mS} \sum_{s=1}^{S} \sum_{i=1}^{m} E(Y_{si} - \mu_{si})^2 \\ &= \frac{1}{mS} \sum_{s=1}^{S} \sum_{i=1}^{m} var(Y_{si}) \\ &= \sigma^2. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} E(\hat{\sigma}_0^2) &= \frac{1}{\binom{m}{2}S}\sum_{s=1}^{S}\sum_{i=1}^{m-1}\sum_{i'>i}^{m} E(Y_{si}-\mu_{si})(Y_{si'}-\mu_{si'}) \\ &= \frac{1}{\binom{m}{2}S}\sum_{s=1}^{S}\sum_{i=1}^{m-1}\sum_{i'>i}^{m} cov(Y_{si},Y_{si'}) \\ &= \sigma_0^2. \end{aligned}$$

We conclude for the known value of β that $\hat{\sigma}^2$ and $\hat{\sigma}_0^2$ are unbiased moment estimates for σ^2 and σ_0^2, respectively. The expected value of sample location lag one and lag two correlations are approximated as,

$$E(W_2) \cong \frac{\sum_{s=1}^{S-1}\sum_{i=1}^{m}\sum_{i'=1}^{m} E(Y_{si}-\mu_{si})(Y_{(s+1)i'}-\mu_{(s+1)i'})/m^2(S-1)}{\sum_{s=1}^{S}\sum_{i=1}^{m} E(Y_{si}-\mu_{si})^2/mS}$$

$$= \frac{\frac{1}{m^2(S-1)}\sum_{s=1}^{S-1}\sum_{i=1}^{m}\sum_{i'=1}^{m}\frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}\tau\sigma^2}{\sigma^2}$$

$$= \frac{\tau}{S-1}\sum_{s=1}^{S-1}\frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}, \qquad (3.23)$$

and

$$E(W_3) \cong \frac{\sum_{s=1}^{S-2}\sum_{i=1}^{m}\sum_{i'=1}^{m} E(Y_{si}-\mu_{si})(Y_{(s+2)i'}-\mu_{(s+2)i'})/m^2(S-2)}{\sum_{s=1}^{S}\sum_{i=1}^{m} E(Y_{si}-\mu_{si})^2/mS}$$

$$= \frac{\frac{1}{m^2(S-2)}\sum_{s=1}^{S-2}\sum_{i=1}^{m}\sum_{i'=1}^{m}\frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}\tau\sigma^2}{\sigma^2}$$

$$= \frac{\tau}{S-2}\sum_{s=1}^{S-2}\frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}, \tag{3.24}$$

therefore, $\frac{W_2(S-1)}{\sum_{s=1}^{S-1}\frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}}$ and $\frac{W_3(S-2)}{\sum_{s=1}^{S-2}\frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}}$ are approximately unbiased moment estimates of τ based on sample lag one and lag two correlations, respectively.

■

From (3.18) and Lemma 3.2.2, the moment estimators of σ^2_γ using lag one correlation, σ^2_γ using lag two correlation, σ^2_α, and σ^2_ϵ are given by

$$\hat{\sigma}^2_\gamma = \frac{W_2(S-1)}{\sum_{s=1}^{S-1}\frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}}\hat{\sigma}^2, \quad \text{by using lag one correlation} \tag{3.25}$$

$$\hat{\sigma}^2_\gamma = \frac{W_3(S-2)}{\sum_{s=1}^{S-2}\frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}}\hat{\sigma}^2, \quad \text{by using lag two correlation} \tag{3.26}$$

$$\hat{\sigma}^2_\alpha = \hat{\sigma}^2_0 - \hat{\sigma}^2_\gamma, \tag{3.27}$$

and

$$\hat{\sigma}_\epsilon^2 = \hat{\sigma}^2 - \hat{\sigma}_0^2, \tag{3.28}$$

respectively.

To estimate the model's scale parameters with independent location random effects, we applied moment estimation approaches, and the results of lag one and lag two correlations are compared. In this case, the vector ξ is reduced to $\xi = (\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2)'$. To estimate the regression coefficients β, we apply the GLS method, and for all the scale parameters, we use the results in Lemma 3.2.2 that are equivalent to the same Newton-Raphson iterative method of moment estimation in (3.16). Here, the derivative matrix P in (3.16) is a 3×3 matrix.

By using the lag one correlation structure, the unbiased estimating equation in (3.9) has three sub-equations of (3.8a), (3.8b), and (3.8d). The derivative matrix P using the lag one correlation is given by

$$\begin{bmatrix} \partial\lambda_1/\partial\sigma_\gamma^2 & \partial\lambda_1/\partial\sigma_\alpha^2 & \partial\lambda_1/\partial\sigma_\epsilon^2 \\ \partial\lambda_2/\partial\sigma_\gamma^2 & \partial\lambda_2/\partial\sigma_\alpha^2 & \partial\lambda_2/\partial\sigma_\epsilon^2 \\ \partial\lambda_4/\partial\sigma_\gamma^2 & \partial\lambda_4/\partial\sigma_\alpha^2 & \partial\lambda_4/\partial\sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{(\sigma_\alpha^2+\sigma_\epsilon^2)\lambda_{21}}{\lambda_1^2} & \frac{-\sigma_\gamma^2\lambda_{21}}{\lambda_1^2} & \frac{-\sigma_\gamma^2\lambda_{21}}{\lambda_1^2} \\ 1 & 1 & 0 \end{bmatrix}, \tag{3.29}$$

where from Lemma 3.2.2

$$\lambda_1 = E(W_1) = \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\epsilon^2 = \sigma^2, \tag{3.30}$$

$$\lambda_2 = E(W_2) = \frac{\tau}{S-1} \sum_{s=1}^{S-1} \frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}, \tag{3.31}$$

$$\lambda_4 = E(W_4) = \sigma_\gamma^2 + \sigma_\alpha^2 = \sigma_0^2, \tag{3.32}$$

$$\lambda_{21} = \frac{1}{S-1} \sum_{s=1}^{S-1} \frac{n^*_{s,s+1}}{\sqrt{n_s n_{s+1}}}. \tag{3.33}$$

Similarly, the unbiased estimating equation in (3.9) using the lag two correlation has three sub-equations of (3.8a), (3.8c), and (3.8d). The respective derivative matrix P is given by

$$\begin{bmatrix} \partial\lambda_1/\partial\sigma_\gamma^2 & \partial\lambda_1/\partial\sigma_\alpha^2 & \partial\lambda_1/\partial\sigma_\epsilon^2 \\ \partial\lambda_3/\partial\sigma_\gamma^2 & \partial\lambda_3/\partial\sigma_\alpha^2 & \partial\lambda_3/\partial\sigma_\epsilon^2 \\ \partial\lambda_4/\partial\sigma_\gamma^2 & \partial\lambda_4/\partial\sigma_\alpha^2 & \partial\lambda_4/\partial\sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{(\sigma_\alpha^2+\sigma_\epsilon^2)\lambda_{31}}{\lambda_1^2} & \frac{-\sigma_\gamma^2\lambda_{31}}{\lambda_1^2} & \frac{-\sigma_\gamma^2\lambda_{31}}{\lambda_1^2} \\ 1 & 1 & 0 \end{bmatrix}, \tag{3.34}$$

where from Lemma 3.2.2

$$\lambda_3 = E(W_3) = \frac{\tau}{S-2} \sum_{s=1}^{S-2} \frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}, \tag{3.35}$$

$$\lambda_{31} = \frac{1}{S-2} \sum_{s=1}^{S-2} \frac{n^*_{s,s+2}}{\sqrt{n_s n_{s+2}}}. \tag{3.36}$$

3.2.2.4 Computational Steps

Recall from (3.5) that the GLS method provides a consistent and efficient estimate for β. The MM estimates of σ_γ^2, σ_α^2, and σ_ϵ^2, can be obtained from equations (3.25)(or (3.26)), (3.27), and (3.28), respectively. The following steps are performed to obtain a suitable estimate for each scale parameter.

Step 1: Starting with an appropriate set of initial values for the scale parameters as $\xi^{(0)} = (\sigma_\gamma^{2\,(0)}, \sigma_\alpha^{2\,(0)}, \sigma_\epsilon^{2\,(0)})'$, we construct Σ from (3.18) and then estimate β (say $\beta^{(0)}$) by using (3.5).

Step 2: With gained $\beta^{(0)}$ in Step 1, we update the values of σ_γ^2, σ_α^2, and σ_ϵ^2 by using the results of Lemma 3.2.2. This new set of scale parameters is called $\xi^{(1)} = (\sigma_\gamma^{2\,(1)}, \sigma_\alpha^{2\,(1)}, \sigma_\epsilon^{2\,(1)})'$.

Step 3: Now we use these new values of scale parameters from Step 2 in Step 1 to update β (say $\beta^{(1)}$). Then $\beta^{(1)}$ is used in Step 2 to improve ξ. We iterate the above cycle to reach convergence.

3.3 A Simulation Study

To examine the behaviour of GLS estimate of β and moment estimate of scale and correlation parameters, several simulation studies have been conducted. We choose a sequence of $S = 100$ equi-spaced locations with two observations at each location ($m = 2$). Note that the distance between any two adjacent locations is equal to unit one, and the user-specified distance to create a cluster of locations is assumed to be equal 4, that is, $d = 4$. Recall from (2.6) that y_{si} the response of the i^{th} family member at the s^{th} location is influenced by some fixed covariates, a vector of location random effects from neighbouring locations within the user-specified distance $d = 4$, and a common family random effect. The linear model is given by

$$\begin{aligned} y_{si} &= x'_{si}\beta + \omega'_s\tilde{\gamma}_s + \alpha_s + \epsilon_{si} \\ &= x'_{si}\beta + \frac{1}{\sqrt{n_s}}1'_{n_s}\tilde{\gamma}_s + \alpha_s + \epsilon_{si} \\ &= x'_{si}\beta + \frac{1}{\sqrt{n_s}}\sum_{j=1}^{n_s}\gamma_{sj} + \alpha_s + \epsilon_{si}, \end{aligned} \tag{3.37}$$

where $x_{si} = (x_{si1}, ..., x_{sip})'$ is a 4-dimensional fixed-covariate vector associated with the locations and family members. In this simulation study, we choose fixed covariates as follows:

- Let $x_{si1} = 1$ be an intercept covariate for $s = 1, 2, ..., S$ and $i = 1, 2, ..., m$
- Similar to the fixed epidemiological covariate used by Mariathas and Sutradhar (2016), we define x_{si2} as shown in equation (3.38):

$$x_{si2} = \begin{cases} 0, & \text{if } 1 \leqslant s \leqslant S/8, \text{ (locations are on high ground)} \\ 1, & \text{if } S/8 + 1 \leqslant s \leqslant 3S/8, \text{ (on plane ground)} \\ 0, & \text{if } 3S/8 + 1 \leqslant s \leqslant S, \text{ (on high ground).} \end{cases} \tag{3.38}$$

- Let x_{si3} be the continuous uniform covariate associated with the age of individuals

$$x_{si3} \sim unif(10, 30). \tag{3.39}$$

- Let x_{si4} be a fixed binary covariate related to the gender of individuals such that

$$x_{si4} = \begin{cases} 0, & \text{if the family member is male} \\ 1, & \text{if the family member is female.} \end{cases} \quad (3.40)$$

For the regression coefficients, we chose two sets of $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)'$,

$$\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' \equiv (0.3, -0.5, 0.2, 0.5)', \quad (3.41)$$

$$\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' \equiv (0.5, 0, 0.8, 0.2)'. \quad (3.42)$$

In practice, $\beta_2 = -0.5$ indicates negative effect of plane ground on the response variable.

3.3.1 Selection of Variance Components

As mentioned in the previous chapter, the vector of location random effect belonging to cluster C_s denoted by $\tilde{\gamma}_s = (\gamma_{s1}, \gamma_{s2}, ..., \gamma_{sj}, ..., \gamma_{sn_s})'$ is a subset of the vector of all S location random effects denoted by $\gamma = (\gamma_1^*, \gamma_2^*, ..., \gamma_S^*)'$. In the present setup, for example, the number of location random effects within C_2 is $n_2 = 4$ with $\tilde{\gamma}_2 = (\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24})' = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)'$. In the case of the family random effect, all family members at a particular location are affected by a shared latent variable independent of other latent variables from other families. The latent variable (say family random effect) at location s is denoted by α_s. Then the family random effect follows

$$\alpha_s \overset{iid}{\sim} N(0, \sigma_\alpha^2). \quad (3.43)$$

Furthermore, for the location random effect,

- if the location random effects are independent ($\phi = 0$), then

$$\gamma_s^* \overset{iid}{\sim} N(0, \sigma_\gamma^2), \tag{3.44}$$

- and if the location random effects ($\phi \neq 0$) are equi-correlated, then

$$corr(\gamma_w^*, \gamma_s^*) = \begin{cases} 1, & \text{for } d_{ws} = 0 \\ \phi, & \text{for } 0 < d_{ws} \leqslant d \\ 0, & \text{for } d_{ws} > d \end{cases} \tag{3.45}$$

$$\gamma_s^* \sim N(0, \sigma_\gamma^2). \tag{3.46}$$

For the error model, we have

$$\epsilon_{si} \overset{iid}{\sim} N(0, \sigma_\epsilon^2). \tag{3.47}$$

It is important to consider different values of σ_γ^2, σ_α^2, and σ_ϵ^2 to assess the estimation behaviour of these parameters. Therefore, we select

$$\sigma_\gamma^2 \equiv (0.1, 0.5, 1.0), \tag{3.48}$$

$$\sigma_\alpha^2 \equiv (0.2, 0.5), \tag{3.49}$$

$$\sigma_\epsilon^2 \equiv (0.25, 1.00). \tag{3.50}$$

In practice, it is essential to evaluate the estimate of parameters for the small and large true values of scale parameters. It helps us to understand which set of parameters presents better estimates. We consider the simulation studies under two scenarios, first for independent location random effects ($\phi = 0$), and second for equi-correlated location random effects ($\phi = 0.3$). These simulation studies were conducted using R software.

3.3.2 The Number of Location Random Effects in Different Regions of Two Neighbouring Clusters

In the first scenario with independent location random effects, the unbiased estimating equations related to scale parameters are functions of known values of n_s, n_w, and n^*_{ws}. In contrast, in the second scenario with equi-correlated location random effects, the unbiased estimating equations related to the scale and correlation parameters are functions of n_s, n_w, n^*_{ws}, n^*_s, n^*_w, and n_{ws}. As discussed in Section 2.3, specific patterns and formulas for n_w, n^*_{ws}, n_{ws} were obtained for three values of $d = 2, 4$, and 6. Values of n_w and n^*_{ws} for $d = 4$ were given in Tables 2.1 and 2.4, respectively. Moreover, equation (2.26) calculated the value of correlated uncommon pairs of n_{ws} when $d = 4$. For illustrative purposes, we display the values of n_w, n_s, n^*_{ws}, $n^*_{w(2)}$, $n^*_{s(2)}$, and n_{ws} for different pairs of w and s in Table 3.1.

Table 3.1: Values of n_w, n_s, n^*_{ws}, $n^*_{w(2)}$, $n^*_{s(2)}$, and n_{ws} for selected pairs of locations with $S = 100$ and $d = 4$.

Pairs of locations (w, s)	n_w	n_s	n^*_{ws}	$n^*_{w(2)}$	$n^*_{s(2)}$	n_{ws}
(1,2)	3	4	3	0	1	0
(1,3)	3	5	3	0	1	0
(1,4)	3	5	2	1	2	2
(1,9)	3	5	0	1	1	1
(2,4)	4	5	3	1	1	1
(2,7)	4	5	0	4	4	10
(2,9)	4	5	0	2	2	3
(3,5)	5	5	3	1	1	1
(3,7)	5	5	1	3	3	6
(3,9)	5	5	0	3	3	6
(50,55)	5	5	0	4	4	10
(50,56)	5	5	0	3	3	6
(50,58)	5	5	0	1	1	1
(94,98)	5	5	1	3	3	6
(94,99)	5	4	0	4	4	10
(94,100)	5	3	0	3	3	6
(98,99)	5	4	4	0	0	0
(97,100)	5	3	2	2	1	2
(98,100)	5	3	3	1	0	0
(99,100)	4	3	3	1	0	0

3.3.3 Simulation Results When Location Random Effects are Independent

We begin by using the proposed model (3.37) with (3.44), (3.46), (3.47), and selected values of scale parameters to generate y_{si}. The GLS and MM estimating equations in Section 3.2.2.3 were then used to iteratively compute estimates of the model parameters. This process was repeated 500 times and we set the initial values of the parameters to $\sigma_{\gamma}^{2(0)} = 0.01$, $\sigma_{\alpha}^{2(0)} = 0.01$, and $\sigma_{\epsilon}^{2(0)} = 0.01$ and follow the computational steps in Section 3.2.2.4 to obtain estimates of the parameters up to 10^{-3} level of accuracy. The simulated means (SMs) along with their simulated standard errors (SSEs) were shown from Table 3.2 to Table 3.9. The MM variance estimates from Table 3.2 to Table 3.5 are based on the lag one correlation statistic, whereas those from Table 3.6 to Table 3.9 are based on the lag two correlation statistic. Each of the following tables summarizes the simulation results for different true values of parameters as below.

- Table 3.2 and Table 3.6 present the estimates of $\beta = (0.3, -0.5, 0.2, 0.5)'$, $\sigma_{\epsilon}^2 = 0.25$, $\sigma_{\gamma}^2 \equiv (0.1, 0.5, 1.0)$, and $\sigma_{\alpha}^2 \equiv (0.2, 0.5)$ based on lag one and lag two correlation statistics, respectively.

- Table 3.3 and Table 3.7 present the estimates of $\beta = (0.5, 0, 0.8, 0.2)'$, $\sigma_{\epsilon}^2 = 0.25$, $\sigma_{\gamma}^2 \equiv (0.1, 0.5, 1.0)$, and $\sigma_{\alpha}^2 \equiv (0.2, 0.5)$ based on lag one and lag two correlation statistics, respectively.

- Table 3.4 and Table 3.8 present the estimates of $\beta = (0.3, -0.5, 0.2, 0.5)'$, $\sigma_{\epsilon}^2 = 1.00$, $\sigma_{\gamma}^2 \equiv (0.1, 0.5, 1.0)$, and $\sigma_{\alpha}^2 \equiv (0.2, 0.5)$ based on lag one and lag two correlation statistics, respectively.

- Table 3.5 and Table 3.9 present the estimates of $\beta = (0.5, 0, 0.8, 0.2)'$, $\sigma_{\epsilon}^2 = 1.00$, $\sigma_{\gamma}^2 \equiv (0.1, 0.5, 1.0)$, and $\sigma_{\alpha}^2 \equiv (0.2, 0.5)$ based on lag one and lag two correlation statistics, respectively.

As far as the GLS regression parameter estimation is concerned, Tables 3.2 to 3.9 show similar results for two different sets of β. Based on both lag one and lag two correlation statistics, regression parameters estimates provide satisfactory results. For example from Table 3.2, when the true values of parameters were $\beta = (0.3, -0.5, 0.2, 0.5)'$ the GLS estimates of β based on lag one correlation statistics with $\sigma^2_\epsilon = 0.25$, $\sigma^2_\gamma = 0.1$, and $\sigma^2_\alpha = 0.2$ is $\hat{\beta} = (0.2929, -0.4784, 0.1999, 0.5000)'$. Similarly from Table 3.6, the GLS estimates of β based on lag two correlation statistics is $\hat{\beta} = (0.2899, -0.4782, 0.2000, 0.4990)'$.

Also from Table 3.5, when the true values of the parameters were $\beta = (0.5, 0, 0.8, 0.2)'$, $\sigma^2_\epsilon = 1.00$, $\sigma^2_\gamma = 0.1$, and $\sigma^2_\alpha = 0.2$, the GLS estimates of regression components with lag one correlation were $\hat{\beta} = (0.4921, 0.0100, 0.8004, 0.1960)'$. In addition from Table 3.9, with the same true values of parameters, the regression components based on lag two correlation were found to be $\hat{\beta} = (0.4945, 0.0075, 0.8000, 0.2044)'$.

Regarding the method of moment estimation of scale parameters, the estimates of σ^2_ϵ and σ^2_α are unbiased for both lag one and lag two correlations when the true values of variance components are small. However, as the true values of σ^2_γ and σ^2_α increase, the estimates become slightly biased. For example the results from Table 3.3 show that when the true values were $\sigma^2_\epsilon = 0.25$, $\sigma^2_\gamma = 0.1$, and $\sigma^2_\alpha = 0.2$ the MM estimates are $\hat{\sigma}^2_\epsilon = 0.2450$, $\hat{\sigma}^2_\gamma = 0.0875$, and $\hat{\sigma}^2_\alpha = 0.2013$. However, from Table 3.7 when the true values were $\sigma^2_\epsilon = 0.25$, $\sigma^2_\gamma = 1.00$, and $\sigma^2_\alpha = 0.5$ the MM estimates are $\hat{\sigma}^2_\epsilon = 0.2458$, $\hat{\sigma}^2_\gamma = 0.7935$, and $\hat{\sigma}^2_\alpha = 0.5782$.

Overall, based on the lag one correlation statistic, the simulation results provide better estimates for the scale parameters.

Table 3.2: GLS estimates of regression coefficients and lag one correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.3, -0.5, 0.lemma2, 0.5)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	0.1	0.2	SM	0.2929	-0.4784	0.1999	0.5000
			SSE	(0.0089)	(0.0092)	(0.0003)	(0.0042)
	0.5	0.2	SM	0.3031	-0.4634	0.1991	0.5003
			SSE	(0.0119)	(0.0155)	(0.0004)	(0.0040)
	1.00	0.2	SM	0.2734	-0.4503	0.2005	0.4969
			SSE	(0.0137)	(0.0199)	(0.0004)	(0.0040)
	0.1	0.5	SM	0.2936	-0.4779	0.1999	0.4963
			SSE	(0.0095)	(0.0109)	(0.0003)	(0.0044)
	0.5	0.5	SM	0.2860	-0.4576	0.1998	0.5027
			SSE	(0.0116)	(0.0171)	(0.0004)	(0.0041)
	1.00	0.5	SM	0.3011	-0.4514	0.1989	0.5001
			SSE	(0.0151)	(0.0218)	(0.0004)	(0.0042)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
0.25	0.1	0.2	SM	0.2470	0.0779	0.2039
			SSE	(0.0026)	(0.0028)	(0.0016)
	0.5	0.2	SM	0.2462	0.4239	0.2138
			SSE	(0.0016)	(0.0073)	(0.0035)
	1.00	0.2	SM	0.2460	0.8565	0.2262
			SSE	(0.0016)	(0.0133)	(0.0043)
	0.1	0.5	SM	0.2464	0.0698	0.5081
			SSE	(0.0016)	(0.0042)	(0.0052)
	0.5	0.5	SM	0.2457	0.4178	0.5140
			SSE	(0.0016)	(0.0084)	(0.0057)
	1.00	0.5	SM	0.2457	0.8482	0.5283
			SSE	(0.0016)	(0.0145)	(0.0067)

Table 3.3: GLS estimates of regression coefficients and lag one correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.5, 0, 0.8, 0.2)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	0.1	0.2	SM	0.5064	-0.0013	0.8001	0.1960
			SSE	(0.0085)	(0.0095)	(0.0004)	(0.0041)
	0.5	0.2	SM	0.5018	-0.0118	0.8001	0.1985
			SSE	(0.01)	(0.0155)	(0.0003)	(0.0040)
	1.00	0.2	SM	0.4730	0.0370	0.8002	0.2031
			SSE	(0.0123)	(0.0198)	(0.0003)	(0.0430)
	0.1	0.5	SM	0.4780	-0.0104	0.8006	0.1986
			SSE	(0.0089)	(0.0112)	(0.0003)	(0.0037)
	0.5	0.5	SM	0.4883	0.0006	0.8004	0.2055
			SSE	(0.0114)	(0.0165)	(0.0004)	(0.0042)
	1.00	0.5	SM	0.4922	-0.0230	0.8003	0.1992
			SSE	(0.0136)	(0.0220)	(0.0003)	(0.0045)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
0.25	0.1	0.2	SM	0.2450	0.0875	0.2013
			SSE	(0.0015)	(0.0024)	(0.0028)
	0.5	0.2	SM	0.2480	0.4450	0.2110
			SSE	(0.0016)	(0.0081)	(0.0035)
	1.00	0.2	SM	0.2460	0.8640	0.2340
			SSE	(0.0016)	(0.0140)	(0.0046)
	0.1	0.5	SM	0.2464	0.1080	0.4730
			SSE	(0.0015)	(0.0035)	(0.0050)
	0.5	0.5	SM	0.2477	0.4359	0.5180
			SSE	(0.0016)	(0.0093)	(0.0063)
	1.00	0.5	SM	0.2480	0.8816	0.5290
			SSE	(0.0016)	(0.0160)	(0.0073)

Table 3.4: GLS estimates of regression coefficients and lag one correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.3, -0.5, 0.2, 0.5)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
1.00	0.1	0.2	SM	0.3102	-0.5028	0.1995	0.4951
			SSE	(0.0142)	(0.0119)	(0.0006)	(0.0066)
	0.5	0.2	SM	0.2683	-0.4522	0.2010	0.4841
			SSE	(0.0169)	(0.0181)	(0.0006)	(0.0073)
	1.00	0.2	SM	0.2508	-0.4341	0.2015	0.5034
			SSE	(0.0173)	(0.0226)	(0.0006)	(0.0074)
	0.1	0.5	SM	0.2549	-0.4443	0.1999	0.5138
			SSE	(0.0322)	(0.0288)	(0.0014)	(0.0193)
	0.5	0.5	SM	0.2728	-0.4502	0.2003	0.5078
			SSE	(0.0173)	(0.0186)	(0.0007)	(0.0077)
	1.00	0.5	SM	0.2518	-0.4422	0.2015	0.4978
			SSE	(0.0183)	(0.0233)	(0.0007)	(0.0076)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
1.00	0.1	0.2	SM	0.8228	0.1410	0.3564
			SSE	(0.0042)	(0.0042)	(0.0048)
	0.5	0.2	SM	0.9314	0.4544	0.2655
			SSE	(0.0053)	(0.0086)	(0.0063)
	1.00	0.2	SM	0.9693	0.8406	0.2608
			SSE	(0.0061)	(0.0140)	(0.0069)
	0.1	0.5	SM	0.9847	0.0498	0.5162
			SSE	(0.0053)	(0.0044)	(0.0066)
	0.5	0.5	SM	0.97702	0.4261	0.5234
			SSE	(0.0062)	(0.0097)	(0.0088)
	1.00	0.5	SM	0.9855	0.8383	0.5362
			SSE	(0.0065)	(0.0161)	(0.0097)

Table 3.5: GLS estimates of regression coefficients and lag one correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.5, 0, 0.8, 0.2)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
1.00	0.1	0.2	SM	0.4921	0.0100	0.8004	0.1960
			SSE	(0.0131)	(0.0117)	(0.0006)	(0.0070)
	0.5	0.2	SM	0.5092	-0.0213	0.7999	0.1933
			SSE	(0.0169)	(0.0178)	(0.0006)	(0.0073)
	1.00	0.2	SM	0.5125	0.0070	0.7985	0.1996
			SSE	(0.017)	(0.0211)	(0.0006)	(0.0075)
	0.1	0.5	SM	0.5132	0.0017	0.7997	0.1900
			SSE	(0.0160)	(0.0135)	(0.0007)	(0.0076)
	0.5	0.5	SM	0.4954	-0.0039	0.7996	0.1982
			SSE	(0.0172)	(0.0184)	(0.0007)	(0.0008)
	1.00	0.5	SM	0.4960	0.0018	0.8001	0.1959
			SSE	(0.0176)	(0.0242)	(0.0006)	(0.0078)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
1.00	0.1	0.2	SM	0.8210	0.1430	0.3560
			SSE	(0.0043)	(0.0042)	(0.0045)
	0.5	0.2	SM	0.9396	0.4775	0.2600
			SSE	(0.0055)	(0.0090)	(0.0066)
	1.00	0.2	SM	0.9780	0.8530	0.2630
			SSE	(0.0059)	(0.0154)	(0.0070)
	0.1	0.5	SM	0.9244	0.1425	0.5230
			SSE	(0.0053)	(0.0048)	(0.0067)
	0.5	0.5	SM	0.9670	0.4400	0.5180
			SSE	(0.0060)	(0.0100)	(0.0090)
	1.00	0.5	SM	0.9892	0.8506	0.5490
			SSE	(0.0065)	(0.0172)	(0.0107)

Table 3.6: GLS estimates of regression coefficients and lag two correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.3, -0.5, 0.2, 0.5)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	0.1	0.2	SM	0.2899	-0.4782	0.2000	0.4990
			SSE	(0.0081)	(0.0092)	(0.0003)	(0.0039)
	0.5	0.2	SM	0.2916	-0.4607	0.1997	0.4987
			SSE	(0.0114)	(0.0156)	(0.0004)	(0.0041)
	1.00	0.2	SM	0.2838	-0.4436	0.1998	0.5075
			SSE	(0.0131)	(0.0199)	(0.0003)	(0.0042)
	0.1	0.5	SM	0.2947	-0.4785	0.1996	0.5051
			SSE	(0.0091)	(0.0110)	(0.0004)	(0.0041)
	0.5	0.5	SM	0.3008	-0.4592	0.1990	0.5013
			SSE	(0.0128)	(0.0173)	(0.0004)	(0.0043)
	1.00	0.5	SM	0.2951	-0.5013	0.2003	0.4990
			SSE	(0.0140)	(0.0225)	(0.0004)	(0.0044)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
0.25	0.1	0.2	SM	0.2467	0.0778	0.2059
			SSE	(0.0016)	(0.0027)	(0.0031)
	0.5	0.2	SM	0.2461	0.3917	0.2444
			SSE	(0.0016)	(0.0081)	(0.0048)
	1.00	0.2	SM	0.2454	0.7720	0.2917
			SSE	(0.0016)	(0.0134)	(0.0067)
	0.1	0.5	SM	0.2460	0.0880	0.4899
			SSE	(0.0016)	(0.0037)	(0.0051)
	0.5	0.5	SM	0.2485	0.4085	0.5344
			SSE	(0.0015)	(0.0110)	(0.0077)
	1.00	0.5	SM	0.2457	0.8040	0.5840
			SSE	(0.0016)	(0.0166)	(0.0100)

Table 3.7: GLS estimates of regression coefficients and lag two correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.5, 0, 0.8, 0.2)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	0.1	0.2	SM	0.4871	0.0217	0.8002	0.1985
			SSE	(0.0083)	(0.0092)	(0.0003)	(0.0040)
	0.5	0.2	SM	0.4880	0.0374	0.7998	0.2024
			SSE	(0.0100)	(0.0156)	(0.0003)	(0.0042)
	1.00	0.2	SM	0.4838	0.0570	0.7998	0.2075
			SSE	(0.0131)	(0.0200)	(0.0003)	(0.0042)
	0.1	0.5	SM	0.4850	0.0200	0.8002	0.2015
			SSE	(0.0091)	(0.0110)	(0.0004)	(0.0044)
	0.5	0.5	SM	0.5008	0.0408	0.7990	0.2013
			SSE	(0.0128)	(0.0173)	(0.0004)	(0.0043)
	1.00	0.5	SM	0.4815	0.0533	0.7998	0.2012
			SSE	(0.0137)	(0.0221)	(0.0004)	(0.0044)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
0.25	0.1	0.2	SM	0.2466	0.0775	0.2056
			SSE	(0.0017)	(0.0029)	(0.0041)
	0.5	0.2	SM	0.2460	0.3930	0.2440
			SSE	(0.0016)	(0.0080)	(0.0050)
	1.00	0.2	SM	0.2453	0.7720	0.2940
			SSE	(0.0016)	(0.0134)	(0.0067)
	0.1	0.5	SM	0.2460	0.0900	0.4890
			SSE	(0.0016)	(0.0037)	(0.0051)
	0.5	0.5	SM	0.2456	0.3897	0.5414
			SSE	(0.0016)	(0.0096)	(0.0074)
	1.00	0.5	SM	0.2458	0.7935	0.5782
			SSE	(0.0016)	(0.0164)	(0.0096)

Table 3.8: GLS estimates of regression coefficients and lag two correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.3, -0.5, 0.2, 0.5)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
1.00	0.1	0.2	SM	0.3254	-0.4965	0.1985	0.5059
			SSE	(0.0137)	(0.0122)	(0.0006)	(0.0073)
	0.5	0.2	SM	0.2748	-0.4441	0.2003	0.5033
			SSE	(0.0172)	(0.0181)	(0.0007)	(0.0008)
	1.00	0.2	SM	0.3029	-0.4293	0.1987	0.5029
			SSE	(0.0174)	(0.0223)	(0.0006)	(0.0074)
	0.1	0.5	SM	0.2960	-0.4626	0.1996	0.4935
			SSE	(0.0162)	(0.0135)	(0.0007)	(0.0079)
	0.5	0.5	SM	0.28008	-0.4539	0.2002	0.4964
			SSE	(0.0166)	(0.0187)	(0.0007)	(0.0074)
	1.00	0.5	SM	0.3009	-0.4366	0.1991	0.4883
			SSE	(0.0193)	(0.0233)	(0.0007)	(0.0077)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
1.00	0.1	0.2	SM	0.8639	0.0853	0.3299
			SSE	(0.0045)	(0.0044)	(0.0047)
	0.5	0.2	SM	0.9411	0.3852	0.3124
			SSE	(0.0056)	(0.0098)	(0.0074)
	1.00	0.2	SM	0.9737	0.7416	0.3416
			SSE	(0.0062)	(0.0152)	(0.0091)
	0.1	0.5	SM	0.9539	0.1024	0.5170
			SSE	(0.0058)	(0.0052)	(0.0069)
	0.5	0.5	SM	0.9806	0.3783	0.5579
			SSE	(0.0064)	(0.0113)	(0.0104)
	1.00	0.5	SM	0.9844	0.7643	0.6055
			SSE	(0.0066)	(0.0179)	(0.0124)

Table 3.9: GLS estimates of regression coefficients and lag two correlation moment estimates of scale parameters for the true value of regression parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.5, 0, 0.8, 0.2)'$ and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
1.00	0.1	0.2	SM	0.4945	0.0075	0.8000	0.2044
			SSE	(0.0154)	(0.0121)	(0.0006)	(0.0073)
	0.5	0.2	SM	0.4676	0.0685	0.8009	0.1875
			SSE	(0.0159)	(0.0181)	(0.0006)	(0.0070)
	1.00	0.2	SM	0.4824	0.0753	0.8000	0.1919
			SSE	(0.0178)	(0.0222)	(0.0006)	(0.0074)
	0.1	0.5	SM	0.5020	0.0355	0.7995	0.1889
			SSE	(0.0147)	(0.0132)	(0.0006)	(0.0077)
	0.5	0.5	SM	0.4823	0.0461	0.8003	0.1863
			SSE	(0.0159)	(0.0187)	(0.0006)	(0.0077)
	1.00	0.5	SM	0.4859	0.0580	0.7999	0.1939
			SSE	(0.0186)	(0.0232)	(0.0007)	(0.0076)

(b) Estimations of the scale parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$
1.00	0.1	0.2	SM	0.8633	0.0855	0.3332
			SSE	(0.0046)	(0.0044)	(0.0047)
	0.5	0.2	SM	0.9422	0.3833	0.3161
			SSE	(0.0057)	(0.0097)	(0.0075)
	1.00	0.2	SM	0.9733	0.7477	0.3402
			SSE	(0.0062)	(0.0152)	(0.0090)
	0.1	0.5	SM	0.9560	0.1028	0.5167
			SSE	(0.0058)	(0.0052)	(0.0070)
	0.5	0.5	SM	0.9810	0.3823	0.5540
			SSE	(0.0063)	(0.0114)	(0.0104)
	1.00	0.5	SM	0.9844	0.7650	0.6059
			SSE	(0.0065)	(0.0180)	(0.0125)

3.3.4 Simulation Results for Equi-Correlated Location Random Effects

Once the familial-spatial data y_{si} is generated using (3.37), (3.45), (3.46), and (3.47), we carry out the GLS estimation of β and MM estimation of $\xi = (\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2, \phi)'$ using $\sigma_\gamma^{2(0)} = 0.01$, $\sigma_\alpha^{2(0)} = 0.01$, $\sigma_\epsilon^{2(0)} = 0.01$, and $\phi^{(0)} = 0.01$ as a set of initial values. Note that, in this large familial-spatial set up with $S = 100$ and $m = 2$ and eight parameters, each simulation takes a considerable amount of time. Tables 3.10 to 3.13 show the simulated means and simulated standard errors of GLS estimation of regression parameters and MM estimation of scale and correlation parameters obtained from 500 simulations. Irrespective of the size of the true values of σ_γ^2, σ_α^2, and σ_ϵ^2, the GLS approach performs very well for the estimation of β. For instance, from Table 3.10, when the true values of parameters were $\beta = (0.3, -0.5, 0.2, 0.5)'$, $\sigma_\epsilon^2 = 0.25$, $\sigma_\gamma^2 = 0.5$, $\sigma_\alpha^2 = 0.2$, and $\phi = 0.3$, the simulated mean of β is $\hat{\beta} = (0.2761, -0.4723, 0.2000, 0.4994)'$. Similarly, from Table 3.11 when $\sigma_\epsilon^2 = 1.00$, $\sigma_\gamma^2 = 1.00$, $\sigma_\alpha^2 = 0.5$, and $\phi = 0.3$, the simulated mean of β is $\hat{\beta} = (0.2872, -0.5315, 0.1997, 0.5055)'$.

For the MM estimation of $\xi = (\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2, \phi)'$, when the true value of scale parameters are smaller, the simulated means of these parameters are closer to the true values. Also, by increasing the true values of σ_γ^2 and σ_α^2, the simulated standard errors become larger. For example from Table 3.12, when the true values of parameters were $\sigma_\epsilon^2 = 0.25$, $\sigma_\gamma^2 = 0.5$, $\sigma_\alpha^2 = 0.2$, and $\phi = 0.3$, the simulate mean of ξ is $\hat{\xi} = (0.2419, 0.5139, 0.2046, 0.3117)'$ with standard error of $(0.0016, 0.0117, 0.0036, 0.0135)'$. However, from Table 3.13 when $\sigma_\epsilon^2 = 1.00$, $\sigma_\gamma^2 = 1.00$, $\sigma_\alpha^2 = 0.5$, and $\phi = 0.3$, the simulated mean of ξ is $\hat{\xi} = (0.9658, 1.0831, 0.4965, 0.3317)'$ with standard error of $(0.0062, 0.0250, 0.0096, 0.0154)'$.

Table 3.10: GLS estimate of β and MM estimates of $\xi = (\sigma^2_\gamma, \sigma^2_\alpha, \sigma^2_\epsilon, \phi)'$ from 500 simulations with 100 locations and 2 family members and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.3, -0.5, 0.2, 0.5)'$, $\phi = 0.3$, and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	0.1	0.2	SM	0.3052	-0.5015	0.1998	0.4988
			SSE	(0.0098)	(0.0135)	(0.0003)	(0.0038)
	0.5	0.2	SM	0.2761	-0.4723	0.2000	0.4994
			SSE	(0.0153)	(0.0217)	(0.0003)	(0.0039)
	1.00	0.2	SM	0.2895	-0.5270	0.2004	0.4964
			SSE	(0.0196)	(0.0269)	(0.0004)	(0.004)
	0.1	0.5	SM	0.3001	-0.4908	0.1998	0.5036
			SSE	(0.0110)	(0.0148)	(0.0004)	(0.0040)
	0.5	0.5	SM	0.2614	-0.4741	0.2006	0.5035
			SSE	(0.0162)	(0.0251)	(0.0004)	(0.0042)
	1.00	0.5	SM	0.3013	-0.4984	0.1999	0.4931
			SSE	(0.0199)	(0.0307)	(0.0004)	(0.0045)

(b) Estimations of the scale and correlation parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$	$\hat{\phi}$
0.25	0.1	0.2	SM	0.2422	0.1399	0.1810	0.2888
			SSE	(0.0016)	(0.0038)	(0.0025)	(0.0140)
	0.5	0.2	SM	0.2429	0.5105	0.2028	0.3156
			SSE	(0.0016)	(0.0112)	(0.0035)	(0.0139)
	1.00	0.2	SM	0.2428	0.8755	0.2487	0.2624
			SSE	(0.0016)	(0.0171)	(0.0043)	(0.0102)
	0.1	0.5	SM	0.2429	0.1617	0.4609	0.2587
			SSE	(0.0015)	(0.0044)	(0.0044)	(0.0153)
	0.5	0.5	SM	0.2427	0.5809	0.4773	0.3367
			SSE	(0.0015)	(0.0143)	(0.0059)	(0.0189)
	1.00	0.5	SM	0.2435	1.0204	0.5073	0.3185
			SSE	(0.0016)	(0.0218)	(0.0069)	(0.0137)

Table 3.11: GLS estimate of β and MM estimates of $\xi = (\sigma^2_\gamma, \sigma^2_\alpha, \sigma^2_\epsilon, \phi)'$ from 500 simulations with 100 locations and 2 family members and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.3, -0.5, 0.2, 0.5)'$, $\phi = 0.3$, and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
1.00	0.1	0.2	SM	0.2906	-0.4939	0.2001	0.4917
			SSE	(0.0154)	(0.0151)	(0.0007)	(0.0074)
	0.5	0.2	SM	0.2941	-0.4798	0.1992	0.4918
			SSE	(0.0181)	(0.0258)	(0.0006)	(0.0070)
	1.00	0.2	SM	0.2896	-0.5025	0.1994	0.4938
			SSE	(0.0245)	(0.0308)	(0.0007)	(0.0073)
	0.1	0.5	SM	0.2816	-0.4762	0.1994	0.5079
			SSE	(0.0158)	(0.0167)	(0.0006)	(0.0071)
	0.5	0.5	SM	0.2847	-0.4856	0.1999	0.5030
			SSE	(0.0191)	(0.0281)	(0.0006)	(0.0073)
	1.00	0.5	SM	0.2872	-0.5315	0.1997	0.5055
			SSE	(0.0246)	(0.0343)	(0.0007)	(0.0079)

(b) Estimations of the scale and correlation parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_\gamma$	$\hat{\sigma}^2_\alpha$	$\hat{\phi}$
1.00	0.1	0.2	SM	0.8751	0.1989	0.2726	0.2849
			SSE	(0.0047)	(0.0053)	(0.0049)	(0.0225)
	0.5	0.2	SM	0.9519	0.5622	0.2212	0.3243
			SSE	(0.0059)	(0.0132)	(0.0055)	(0.0151)
	1.00	0.2	SM	0.9585	0.9059	0.2788	0.3781
			SSE	(0.0061)	(0.0202)	(0.0063)	(0.0183)
	0.1	0.5	SM	0.9436	0.1961	0.4806	0.2620
			SSE	(0.0058)	(0.0057)	(0.0071)	(0.0149)
	0.5	0.5	SM	0.9718	0.6259	0.4559	0.3221
			SSE	(0.0062)	(0.0159)	(0.0082)	(0.0187)
	1.00	0.5	SM	0.9729	1.0797	0.4929	0.3504
			SSE	(0.0063)	(0.0262)	(0.0093)	(0.0165)

Table 3.12: GLS estimate of β and MM estimates of $\xi = (\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2, \phi)'$ from 500 simulations with 100 locations and 2 family members and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.5, 0, 0.8, 0.2)'$, $\phi = 0.3$, and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ_ϵ^2	σ_γ^2	σ_α^2	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
0.25	0.1	0.2	SM	0.5037	0.0007	0.7999	0.1983
			SSE	(0.0096)	(0.0132)	(0.0003)	(0.0037)
	0.5	0.2	SM	0.4713	0.0287	0.8002	0.2025
			SSE	(0.0145)	(0.0221)	(0.0003)	(0.0042)
	1.00	0.2	SM	0.4989	-0.0128	0.7996	0.1967
			SSE	(0.0195)	(0.0277)	(0.0003)	(0.0044)
	0.1	0.5	SM	0.4904	0.0058	0.8000	0.2017
			SSE	(0.0115)	(0.0142)	(0.0004)	(0.0041)
	0.5	0.5	SM	0.4639	0.0126	0.8007	0.1958
			SSE	(0.0151)	(0.0253)	(0.0004)	(0.0044)
	1.00	0.5	SM	0.4812	0.0011	0.8001	0.1984
			SSE	(0.0213)	(0.0300)	(0.0004)	(0.0040)

(b) Estimations of the scale and correlation parameters

σ_ϵ^2	σ_γ^2	σ_α^2	Quantity	$\hat{\sigma}_\epsilon^2$	$\hat{\sigma}_\gamma^2$	$\hat{\sigma}_\alpha^2$	$\hat{\phi}$
0.25	0.1	0.2	SM	0.2435	0.1451	0.1785	0.2854
			SSE	(0.0016)	(0.0038)	(0.0025)	(0.015)
	0.5	0.2	SM	0.2419	0.5139	0.2046	0.3117
			SSE	(0.0016)	(0.0117)	(0.0036)	(0.0135)
	1.00	0.2	SM	0.2429	0.8969	0.2430	0.2437
			SSE	(0.0016)	(0.0175)	(0.0043)	(0.0096)
	0.1	0.5	SM	0.2431	0.1540	0.4617	0.2813
			SSE	(0.0016)	(0.0044)	(0.0046)	(0.0194)
	0.5	0.5	SM	0.2431	0.5927	0.4710	0.3234
			SSE	(0.0016)	(0.0143)	(0.0058)	(0.0162)
	1.00	0.5	SM	0.2426	1.0570	0.5001	0.3096
			SSE	(0.0016)	(0.0236)	(0.0070)	(0.0138)

Table 3.13: GLS estimate of β and MM estimates of $\xi = (\sigma^2_\gamma, \sigma^2_\alpha, \sigma^2_\epsilon, \phi)'$ from 500 simulations with 100 locations and 2 family members and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)' = (0.5, 0, 0.8, 0.2)'$, $\phi = 0.3$, and selected true value of scale parameters.

(a) Estimations of the regression parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat\beta_1$	$\hat\beta_2$	$\hat\beta_3$	$\hat\beta_4$
1.00	0.1	0.2	SM	0.4650	0.0270	0.8006	0.1987
			SSE	(0.0145)	(0.0148)	(0.0006)	(0.0071)
	0.5	0.2	SM	0.4860	0.0139	0.7999	0.1938
			SSE	(0.0188)	(0.0260)	(0.0006)	(0.0073)
	1.00	0.2	SM	0.4609	-0.0420	0.8004	0.2147
			SSE	(0.0241)	(0.0317)	(0.0006)	(0.0076)
	0.1	0.5	SM	0.5215	0.0226	0.7987	0.2005
			SSE	(0.0151)	(0.0164)	(0.0006)	(0.0082)
	0.5	0.5	SM	0.4696	0.0157	0.8007	0.2040
			SSE	(0.0200)	(0.0290)	(0.0007)	(0.0081)
	1.00	0.5	SM	0.5152	-0.0190	0.7994	0.1781
			SSE	(0.0237)	(0.0342)	(0.0006)	(0.0075)

(b) Estimations of the scale and correlation parameters

σ^2_ϵ	σ^2_γ	σ^2_α	Quantity	$\hat\sigma^2_\epsilon$	$\hat\sigma^2_\gamma$	$\hat\sigma^2_\alpha$	$\hat\phi$
1.00	0.1	0.2	SM	0.8814	0.2050	0.2682	0.2938
			SSE	(0.0048)	(0.0055)	(0.0050)	(0.0165)
	0.5	0.2	SM	0.9558	0.5570	0.2205	0.3298
			SSE	(0.0058)	(0.0126)	(0.0057)	(0.0149)
	1.00	0.2	SM	0.9602	0.9103	0.2740	0.3622
			SSE	(0.0063)	(0.0203)	(0.0062)	(0.0164)
	0.1	0.5	SM	0.9535	0.2022	0.4656	0.2622
			SSE	(0.0056)	(0.0058)	(0.0070)	(0.0151)
	0.5	0.5	SM	0.9734	0.6170	0.4554	0.3700
			SSE	(0.0063)	(0.0159)	(0.0083)	(0.0344)
	1.00	0.5	SM	0.9685	1.0831	0.4965	0.3317
			SSE	(0.0062)	(0.0250)	(0.0096)	(0.0154)

Chapter 4

Estimation of Spatial-Temporal Linear Dynamic Mixed Effect Model

Spatial-temporal data has been of interest to researchers in the area of biology, ecology, meteorology, medicine, transportation, and forestry. In the spatial-temporal setup, observations are collected from different locations sequentially at different time points. For example, in an air pollution study, one may be interested in examining the influence of random effects from adjacent locations on the amount of specific particulate in the air every hour. This chapter extends the cluster-based spatial correlation model of Mariathas and Sutradhar (2016) to the cluster-based spatial-temporal correlation model by using an AR(1) type dynamic model for linear responses.

4.1 Proposed Cluster-Based Spatial-Temporal Linear Dynamic Mixed Effect Model

Suppose at the s^{th} location we observe a time series y_{s1}, y_{s2},..., y_{st_s}. For simplicity, we consider a balanced dynamic model such that at each location, we have the same number of observations at equi-spaced time points, that is, $t_s = m$ for all $s = 1, ..., S$ locations. Let y_{st} be a continuous response measured at the s^{th} location at time point t that follows an autocorrelation structure model as,

$$\begin{aligned} y_{s1} &= x'_{s1}\beta + \omega'_s\tilde{\gamma}_s + \epsilon_{s1}, \\ y_{st} &= x'_{st}\beta + \theta(y_{s,t-1} - x'_{s,t-1}\beta) + \omega'_s\tilde{\gamma}_s + \epsilon_{st} \quad \text{for } t = 2, 3, ..., m, \end{aligned} \tag{4.1}$$

where $x_{st} = (x_{st1}, ..., x_{stp})'$ is a p-dimensional environmental or individual fixed covariate, $\beta = (\beta_1, ..., \beta_p)'$ is the regression effects of x_{st} on y_{st} for all $s = 1, ..., S$ and $t = 1, ..., m$. Note that in (4.1), θ ($|\theta| < 1$) refers to an autoregressive order one AR(1) dynamic dependence parameter that is leading to a temporal correlation between responses, and $\tilde{\gamma}_s = (\gamma_{s1}, ..., \gamma_{sn_s})'$ is a vector of location random effects, which was already defined in (2.29) with weight vector $\omega_s = \frac{1}{\sqrt{n_s}} 1_{n_s}$. Also, for any two distinct locations s and w, $\tilde{\gamma}_s$ and $\tilde{\gamma}_w$ satisfy the assumptions of (2.31)-(2.38). For $i = 1, ..., n_w$ and $j = 1, ..., n_s$, we have

$$corr(\gamma_{wi}, \gamma_{sj}) = \begin{cases} 1 & \text{for } d_{ws} = 0 \\ \phi & \text{for } 0 < d_{ws} \leqslant d \\ 0 & \text{for } d_{ws} > d, \end{cases} \tag{4.2}$$

and

$$\gamma_{sj} \sim (0, \sigma_\gamma^2), \quad \text{and} \quad \tilde{\gamma}_s \sim (0, \Gamma_{ss}), \tag{4.3}$$

where Γ_{ss} is given in (2.31). The error model ϵ_{st} is independent of γ_{sj} and assumed to follow

$$\epsilon_{st} \overset{iid}{\sim} (0, \sigma_\epsilon^2). \tag{4.4}$$

Now, for simplicity of notation, we use the mean deleted variable as follows,

$$\tilde{y}_{st} = y_{st} - x'_{st}\beta. \tag{4.5}$$

Based on the above assumptions we can rewrite the proposed model (4.1) as,

$$\begin{aligned} \tilde{y}_{s1} &= \omega'_s \tilde{\gamma}_s + \epsilon_{s1}, \\ \tilde{y}_{st} &= \theta \tilde{y}_{s,t-1} + \omega'_s \tilde{\gamma}_s + \epsilon_{st} \quad \text{for } t = 2, 3, ..., m. \end{aligned} \tag{4.6}$$

4.2 Basic Properties

By using the properties of the vector of location random effect in the second chapter, we discuss the mean, variance, and covariance of spatial-temporal responses in the following lemmas.

4.2.1 Mean and Variance

Lemma 4.2.1. The mean and variance of the mean deleted variable for spatial-temporal dynamic mixed model are given by

$$E(\tilde{Y}_{st}) = 0, \tag{4.7}$$

$$\begin{aligned} var(\tilde{Y}_{st}) &= \sigma^2_{st} \\ &= \left\{ \sum_{j=0}^{t-1} \theta^j \right\}^2 \omega'_s \Gamma_{ss} \omega_s + \left\{ \sum_{j=0}^{t-1} \theta^{2j} \right\} \sigma^2_\epsilon. \end{aligned} \tag{4.8}$$

and, the recursive variance formula is

$$\begin{aligned} var(\tilde{Y}_{st}) &= \sigma^2_{st} \\ &= \theta^2 \sigma^2_{s(t-1)} + \Big\{ -1 + 2 \sum_{j=0}^{t-1} \theta^j \Big\} \omega'_s \Gamma_{ss} \omega_s + \sigma^2_\epsilon, \end{aligned} \tag{4.9}$$

where $\sigma^2_{s0} = 0$ and from equation (2.31) and Lemma 2.4.1 with $\omega_s = \frac{1}{\sqrt{n_s}} 1_{n_s}$,

$$\omega'_s \Gamma_{ss} \omega_s = \begin{cases} \sigma^2_\gamma & \text{for } \phi = 0 \\ \sigma^2_\gamma [n_s \phi + (1 - \phi)] & \text{for } \phi \neq 0. \end{cases} \tag{4.10}$$

Proof: From (4.1), (4.3), and (4.4), we conclude that $E(Y_{st}) = x'_{st}\beta$ and from (4.5) yielding $E(\tilde{Y}_{st}) = 0$. By rewriting the model (4.6), $var(Y_{st}) = var(\tilde{Y}_{st})$. For all $s = 1, ..., S$, the variance of the response at $t = 1$ is given by

$$\begin{aligned} \sigma^2_{s1} &= E(\tilde{Y}^2_{s1}) \\ &= E\left[(\omega'_s \tilde{\gamma}_s + \epsilon_{s1})^2 \right] \\ &= \omega'_s \Gamma_{ss} \omega_s + \sigma^2_\epsilon. \end{aligned} \tag{4.11}$$

At $t = 2$, we obtain the recursive variance formula as,

$$\begin{aligned}
\sigma^2_{s2} &= E(\tilde{Y}^2_{s2}) \\
&= E\left[(\theta\tilde{Y}_{s1} + \omega'_s\tilde{\gamma}_s + \epsilon_{s2})^2\right] \\
&= E\left[\theta^2\tilde{Y}^2_{s1} + \omega'_s\tilde{\gamma}_s\tilde{\gamma}'_s\omega_s + \epsilon^2_{s2} + 2\theta\tilde{Y}_{s1}\omega'_s\tilde{\gamma}_s + 2\theta\tilde{Y}_{s1}\epsilon_{s2} + 2\omega'_s\tilde{\gamma}_s\epsilon_{s2}\right] \\
&= \theta^2\sigma^2_{s1} + \omega'_s\Gamma_{ss}\omega_s + \sigma^2_\epsilon + 2\theta E\left[(\omega'_s\tilde{\gamma}_s + \epsilon_{s1})\omega'_s\tilde{\gamma}_s\right] \\
&= \theta^2\sigma^2_{s1} + \omega'_s\Gamma_{ss}\omega_s + \sigma^2_\epsilon + 2\theta\omega'_s\Gamma_{ss}\omega_s \\
&= \theta^2\sigma^2_{s1} + (1 + 2\theta)\omega'_s\Gamma_{ss}\omega_s + \sigma^2_\epsilon, \qquad (4.12)
\end{aligned}$$

by replacing equation (4.11) in (4.12), we have

$$\sigma^2_{s2} = (1 + \theta)^2\omega'_s\Gamma_{ss}\omega_s + (1 + \theta^2)\sigma^2_\epsilon. \qquad (4.13)$$

At $t = 3$, the recursive relation variance formula is

$$\begin{aligned}
\sigma^2_{s3} &= E(\tilde{Y}^2_{s3}) \\
&= E\left[(\theta\tilde{Y}_{s2} + \omega'_s\tilde{\gamma}_s + \epsilon_{s3})^2\right] \\
&= E\left[\theta^2\tilde{Y}^2_{s2} + \omega'_s\tilde{\gamma}_s\tilde{\gamma}'_s\omega_s + \epsilon^2_{s3} + 2\theta\tilde{Y}_{s2}\omega'_s\tilde{\gamma}_s + 2\theta\tilde{Y}_{s2}\epsilon_{s3} + 2\omega'_s\tilde{\gamma}_s\epsilon_{s3}\right] \\
&= \theta^2\sigma^2_{s2} + \omega'_s\Gamma_{ss}\omega_s + \sigma^2_\epsilon + 2\theta E\left[(\theta\tilde{Y}_{s1} + \omega'_s\tilde{\gamma}_s + \epsilon_{s2})\omega'_s\tilde{\gamma}_s\right] \\
&= \theta^2\sigma^2_{s2} + \omega'_s\Gamma_{ss}\omega_s + \sigma^2_\epsilon + 2\theta(1 + \theta)\omega'_s\Gamma_{ss}\omega_s \\
&= \theta^2\sigma^2_{s2} + (1 + 2\theta + 2\theta^2)\omega'_s\Gamma_{ss}\omega_s + \sigma^2_\epsilon, \qquad (4.14)
\end{aligned}$$

by replacing equation (4.13) in (4.14), we have

$$\sigma^2_{s3} = (1 + \theta + \theta^2)^2\omega'_s\Gamma_{ss}\omega_s + (1 + \theta^2 + \theta^4)\sigma^2_\epsilon. \qquad (4.15)$$

Similarly, at $t = 4$, we find

$$\begin{aligned}
\sigma_{s4}^2 &= E(\tilde{Y}_{s4}^2) \\
&= E\left[(\theta\tilde{Y}_{s3} + \omega_s'\tilde{\gamma}_s + \epsilon_{s4})^2\right] \\
&= E\left[\theta^2\tilde{Y}_{s3}^2 + \omega_s'\tilde{\gamma}_s\tilde{\gamma}_s'\omega_s + \epsilon_{s4}^2 + 2\theta\tilde{Y}_{s3}\omega_s'\tilde{\gamma}_s + 2\theta\tilde{Y}_{s3}\epsilon_{s4} + 2\omega_s'\tilde{\gamma}_s\epsilon_{s4}\right] \\
&= \theta^2\sigma_{s3}^2 + \omega_s'\Gamma_{ss}\omega_s + \sigma_\epsilon^2 + 2\theta E\left[(\theta\tilde{Y}_{s2} + \omega_s'\tilde{\gamma}_s + \epsilon_{s3})\omega_s'\tilde{\gamma}_s\right] \\
&= \theta^2\sigma_{s3}^2 + \omega_s'\Gamma_{ss}\omega_s + \sigma_\epsilon^2 + 2\theta\left[1 + \theta + \theta^2\right]\omega_s'\Gamma_{ss}\omega_s \\
&= \theta^2\sigma_{s3}^2 + (1 + 2\theta + 2\theta^2 + 2\theta^3)\omega_s'\Gamma_{ss}\omega_s + \sigma_\epsilon^2, \qquad (4.16)
\end{aligned}$$

by replacing equation (4.15) in (4.16), we have

$$\sigma_{s4}^2 = (1 + \theta + \theta^2 + \theta^3)^2\omega_s'\Gamma_{ss}\omega_s + (1 + \theta^2 + \theta^4 + \theta^6)\sigma_\epsilon^2. \qquad (4.17)$$

Now, the repeated substitution method gives a general expression for the recursive variance formula at time point t

$$\begin{aligned}
var(\tilde{Y}_{st}) &= \sigma_{st}^2 \\
&= \theta^2\sigma_{s(t-1)}^2 + \Big\{-1 + 2\sum_{j=0}^{t-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s + \sigma_\epsilon^2, \qquad (4.18)
\end{aligned}$$

where $\sigma_{s0}^2 = 0$. Following on that

$$\begin{aligned}
var(\tilde{Y}_{st}) &= \sigma_{st}^2 \\
&= \left\{\sum_{j=0}^{t-1}\theta^j\right\}^2\omega_s'\Gamma_{ss}\omega_s + \left\{\sum_{j=0}^{t-1}\theta^{2j}\right\}\sigma_\epsilon^2 \\
&= \left[\frac{1-\theta^t}{1-\theta}\right]^2\omega_s'\Gamma_{ss}\omega_s + \left[\frac{1-\theta^{2t}}{1-\theta^2}\right]\sigma_\epsilon^2. \qquad (4.19)
\end{aligned}$$

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Note that the $var(\tilde{Y}_{st})$ in equation (4.19) required to calculate $E(\tilde{Y}_{st}\omega_s'\tilde{\gamma}_s)$ for all $t = 1, ..., m$. Therefore, by using a repeated substitution method, we have

$$E(\tilde{Y}_{st}\omega_s'\tilde{\gamma}_s) = \left\{\sum_{j=0}^{t-1}\theta^j\right\}\omega_s'\Gamma_{ss}\omega_s = \frac{1-\theta^t}{1-\theta}\omega_s'\Gamma_{ss}\omega_s. \tag{4.20}$$

4.2.2 Covariance

The covariance between spatial-temporal responses consists of covariance of responses from the same location at different time points and covariance of the responses from different locations at any time points. The corresponding covariances are given in the following lemmas.

Lemma 4.2.2. The covariance between two mean deleted spatial-temporal variables $\tilde{y}_{sk}$ and $\tilde{y}_{st}$ from the same location s and different time points of k and t ($k \neq t$) is given by

$$\begin{aligned}\sigma_{sk,st} &= cov(\tilde{Y}_{sk}, \tilde{Y}_{st}) \\ &= \left\{\sum_{j=0}^{k-1}\theta^j\right\}\left\{\sum_{j=0}^{t-1}\theta^j\right\}\omega_s'\Gamma_{ss}\omega_s + \theta^{|t-k|}\left\{\sum_{j=0}^{min(k,t)-1}\theta^{2j}\right\}\sigma_\epsilon^2,\end{aligned} \tag{4.21}$$

and, the recursive relation covariance formula is

$$\begin{aligned}\sigma_{sk,st} &= cov(\tilde{Y}_{sk}, \tilde{Y}_{st}) \\ &= \theta\sigma_{sk,s(t-1)} + \omega_s'\Gamma_{ss}\omega_s\sum_{j=0}^{k-1}\theta^j.\end{aligned} \tag{4.22}$$

Proof: To find the covariance between the spatial-temporal responses of the same location at different time points, that is, $\sigma_{sk,st} = cov(\tilde{Y}_{sk}, \tilde{Y}_{st}) = cov(Y_{sk}, Y_{st})$, we use the repeated substitution method. Start with $k = 1$ and $t = 2$, we have

$$\begin{aligned}\sigma_{s1,s2} = cov(\tilde{Y}_{s1}, \tilde{Y}_{s2}) &= E(\tilde{Y}_{s1}\tilde{Y}_{s2}) \\ &= \theta E(\tilde{Y}_{s1}^2) + E(\tilde{Y}_{s1}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s1}^2 + E(\tilde{Y}_{s1}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s1}^2 + \omega_s'\Gamma_{ss}\omega_s, \qquad (4.23)\end{aligned}$$

by replacing equation (4.19) at point $t = 1$ in (4.23), we have

$$\sigma_{s1,s2} = (1+\theta)\omega_s'\Gamma_{ss}\omega_s + \theta\sigma_\epsilon^2. \qquad (4.24)$$

Now, at $k = 1$ and $t = 3$, we find

$$\begin{aligned}\sigma_{s1,s3} = cov(\tilde{Y}_{s1}, \tilde{Y}_{s3}) &= E(\tilde{Y}_{s1}\tilde{Y}_{s3}) \\ &= \theta E(\tilde{Y}_{s1}\tilde{Y}_{s2}) + E(\tilde{Y}_{s1}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s1,s2} + E(\tilde{Y}_{s1}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s1,s2} + \omega_s'\Gamma_{ss}\omega_s, \qquad (4.25)\end{aligned}$$

by replacing equation (4.24) in (4.25), we have

$$\sigma_{s1,s3} = (1+\theta+\theta^2)\omega_s'\Gamma_{ss}\omega_s + \theta^2\sigma_\epsilon^2. \qquad (4.26)$$

Next, at $k = 1$ and $t = 4$, we find

$$\begin{aligned} \sigma_{s1,s4} = cov(\tilde{Y}_{s1}, \tilde{Y}_{s4}) &= E(\tilde{Y}_{s1}\tilde{Y}_{s4}) \\ &= \theta E(\tilde{Y}_{s1}\tilde{Y}_{s3}) + E(\tilde{Y}_{s1}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s1,s3} + E(\tilde{Y}_{s1}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s1,s3} + \omega_s'\Gamma_{ss}\omega_s, \end{aligned} \tag{4.27}$$

by replacing equations (4.26) in (4.27), we have

$$\sigma_{s1,s4} = (1 + \theta + \theta^2 + \theta^3)\omega_s'\Gamma_{ss}\omega_s + \theta^3\sigma_\epsilon^2. \tag{4.28}$$

In general, for $k = 1$ and $t = 2, 3, 4, ...$

$$\sigma_{s1,st} = \Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s + \theta^{t-1}\sigma_\epsilon^2, \tag{4.29}$$

and the recursive relation covariance formula is

$$\sigma_{s1,st} = \theta\sigma_{s1,s(t-1)} + \omega_s'\Gamma_{ss}\omega_s. \tag{4.30}$$

Now, for $k = 2$ and $t = 3$, we have

$$\begin{aligned} \sigma_{s2,s3} = cov(\tilde{Y}_{s2}, \tilde{Y}_{s3}) &= E(\tilde{Y}_{s2}\tilde{Y}_{s3}) \\ &= \theta E(\tilde{Y}_{s2}^2) + E(\tilde{Y}_{s2}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s2}^2 + E(\tilde{Y}_{s2}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s2}^2 + (1+\theta)\omega_s'\Gamma_{ss}\omega_s, \end{aligned} \tag{4.31}$$

by replacing equation (4.19) at point $t = 2$ in (4.31), we have

$$\sigma_{s2,s3} = (1+\theta)(1+\theta+\theta^2)\omega_s'\Gamma_{ss}\omega_s + \theta(1+\theta^2)\sigma_\epsilon^2. \tag{4.32}$$

Next, at $k = 2$ and $t = 4$, we find

$$\begin{aligned} \sigma_{s2,s4} = cov(\tilde{Y}_{s2}, \tilde{Y}_{s4}) &= E(\tilde{Y}_{s2}\tilde{Y}_{s4}) \\ &= \theta E(\tilde{Y}_{s2}\tilde{Y}_{s3}) + E(\tilde{Y}_{s2}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s2,s3} + E(\tilde{Y}_{s2}\omega_s'\tilde{\gamma}_s) \\ &= \theta\sigma_{s2,s3} + (1+\theta)\omega_s'\Gamma_{ss}\omega_s, \end{aligned} \tag{4.33}$$

by replacing equation (4.32) in (4.33), we have

$$\sigma_{s2,s4} = (1+\theta)(1+\theta+\theta^2+\theta^3)\omega_s'\Gamma_{ss}\omega_s + \theta^2(1+\theta^2)\sigma_\epsilon^2. \tag{4.34}$$

In general, for $k = 2$ and $t = 3, 4, ...$

$$\sigma_{s2,st} = (1+\theta)\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s + \theta^{t-2}(1+\theta^2)\sigma_\epsilon^2, \tag{4.35}$$

and the recursive relation covariance formula is

$$\sigma_{s2,st} = \theta\sigma_{s2,s(t-1)} + (1+\theta)\omega_s'\Gamma_{ss}\omega_s. \tag{4.36}$$

Similarly, for $k = 3$ and $t = 4, 5, ...$ we find

$$\sigma_{s3,st} = (1+\theta+\theta^2)\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s + \theta^{t-3}(1+\theta^2+\theta^4)\sigma_\epsilon^2, \tag{4.37}$$

and the recursive relation covariance formula is

$$\sigma_{s3,st} = \theta\sigma_{s3,s(t-1)} + (1+\theta+\theta^2)\omega_s'\Gamma_{ss}\omega_s. \tag{4.38}$$

From (4.29), (4.35), (4.37), and using repeated substitution method we conclude that for $k < t$

$$\sigma_{sk,st} = \Big\{\sum_{j=0}^{k-1}\theta^j\Big\}\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s + \theta^{t-k}\Big\{\sum_{j=0}^{k-1}\theta^{2j}\Big\}\sigma_\epsilon^2, \tag{4.39}$$

in another way, for any $k \neq t$

$$\sigma_{sk,st} = \Big\{\sum_{j=0}^{k-1}\theta^j\Big\}\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s + \theta^{|t-k|}\Big\{\sum_{j=0}^{min(k,t)-1}\theta^{2j}\Big\}\sigma_\epsilon^2. \tag{4.40}$$

From (4.30), (4.36), (4.38) the recursive relation covariance for $t = k+1, k+2, ..., m$ is given by

$$\begin{aligned}\sigma_{sk,st} &= cov(\tilde{Y}_{sk}, \tilde{Y}_{st}) \\ &= \theta\sigma_{sk,s(t-1)} + \Big\{\sum_{j=0}^{k-1}\theta^j\Big\}\omega_s'\Gamma_{ss}\omega_s. \end{aligned}\tag{4.41}$$

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Lemma 4.2.3. The covariance between two spatial-temporal responses $\tilde{y}_{wk}$ and $\tilde{y}_{st}$ from the locations w and s at any time points of $k = 1, ..., m$ and $t = 1, ..., m$ is given by

$$\begin{aligned}\sigma_{wk,st} &= cov(\tilde{Y}_{wk}, \tilde{Y}_{st}) \\ &= \Big\{\sum_{j=0}^{k-1}\theta^j\Big\}\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_w'\Gamma_{ws}\omega_s, \end{aligned}\tag{4.42}$$

and, the recursive relation covariance formula is

$$\begin{aligned}\sigma_{wk,st} &= cov(\tilde{Y}_{wk}, \tilde{Y}_{st}) \\ &= \theta\sigma_{wk,s(t-1)} + \Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega'_w\Gamma_{ws}\omega_s, \end{aligned} \tag{4.43}$$

where from equation (2.33) and Lemma 2.4.3 with $\omega_w = \frac{1}{\sqrt{n_w}}1_{n_w}$ and $\omega_s = \frac{1}{\sqrt{n_s}}1_{n_s}$,

$$\omega'_w\Gamma_{ws}\omega_s = \begin{cases}\sigma^2_\gamma\frac{n^*_{ws}}{\sqrt{n_w n_s}}, & \text{for } \phi = 0\\ \frac{\sigma^2_\gamma}{\sqrt{n_w n_s}}\Big[n^*_{ws} + \phi\Big(n_{ws} + n^*_{ws}(n^*_w + n^*_s) + n^*_{ws}(n^*_{ws} - 1)\Big)\Big], & \text{for } \phi \neq 0.\end{cases} \tag{4.44}$$

Proof: To derive the covariance formula in (4.42), we use the repeated substitution method. Start with $k = 1$ and $t = 1$, we have

$$\begin{aligned}\sigma_{w1,s1} = cov(\tilde{Y}_{w1}, \tilde{Y}_{s1}) &= E(\tilde{Y}_{w1}\tilde{Y}_{s1}) \\ &= E\left[(\tilde{Y}_{w1}(\omega'_s\tilde{\gamma}_s + \epsilon_{s1})\right] \\ &= E(\tilde{Y}_{w1}\omega'_s\tilde{\gamma}_s) \\ &= \omega'_w\Gamma_{ws}\omega_s. \end{aligned} \tag{4.45}$$

At $k = 1$ and $t = 2$, we have

$$\begin{aligned}\sigma_{w1,s2} = cov(\tilde{Y}_{w1}, \tilde{Y}_{s2}) &= E(\tilde{Y}_{w1}\tilde{Y}_{s2}) \\ &= E\left[(\tilde{Y}_{w1}(\theta\tilde{Y}_{s1} + \omega'_s\tilde{\gamma}_s + \epsilon_{s2})\right] \\ &= \theta\sigma_{w1,s1} + E(\tilde{Y}_{w1}\omega'_s\tilde{\gamma}_s) \\ &= \theta\sigma_{w1,s1} + \omega'_w\Gamma_{ws}\omega_s \\ &= (1+\theta)\omega'_w\Gamma_{ws}\omega_s. \end{aligned} \tag{4.46}$$

Next, at $k = 1$ and $t = 3$, we have

$$\begin{aligned}
\sigma_{w1,s3} = cov(\tilde{Y}_{w1}, \tilde{Y}_{s3}) &= E(\tilde{Y}_{w1}\tilde{Y}_{s3}) \\
&= E\left[(\tilde{Y}_{w1}(\theta\tilde{Y}_{s2} + \omega_s'\tilde{\gamma}_s + \epsilon_{s3})\right] \\
&= \theta\sigma_{w1,s2} + E(\tilde{Y}_{w1}\omega_s'\tilde{\gamma}_s) \\
&= \theta\sigma_{w1,s2} + \omega_w'\Gamma_{ws}\omega_s \\
&= (1 + \theta + \theta^2)\omega_w'\Gamma_{ws}\omega_s. \qquad (4.47)
\end{aligned}$$

In general, for $k = 1$ and $t = 1, 2, ..., m$, we conclude that

$$\begin{aligned}
\sigma_{w1,st} &= cov(\tilde{Y}_{w1}, \tilde{Y}_{st}) \\
&= \theta\sigma_{w1,s(t-1)} + \omega_w'\Gamma_{ws}\omega_s \\
&= \left\{\sum_{j=0}^{t-1}\theta^j\right\}\omega_w'\Gamma_{ws}\omega_s. \qquad (4.48)
\end{aligned}$$

Now, for $k = 2$ and $t = 1$, we find

$$\begin{aligned}
\sigma_{w2,s1} = cov(\tilde{Y}_{w2}, \tilde{Y}_{s1}) &= E(\tilde{Y}_{w2}\tilde{Y}_{s1}) \\
&= E\left[(\tilde{Y}_{w2}(\omega_s'\tilde{\gamma}_s + \epsilon_{s1})\right] \\
&= E(\tilde{Y}_{w2}\omega_s'\tilde{\gamma}_s) \\
&= \theta E(\tilde{Y}_{w1}\omega_s'\tilde{\gamma}_s) + \omega_w'\Gamma_{ws}\omega_s \\
&= \theta\sigma_{w1,s1} + \sigma_{w1,s1} \\
&= (1 + \theta)\omega_w'\Gamma_{ws}\omega_s. \qquad (4.49)
\end{aligned}$$

At $k = 2$ and $t = 2$,

$$\begin{aligned}\sigma_{w2,s2} = cov(\tilde{Y}_{w2}, \tilde{Y}_{s2}) &= E(\tilde{Y}_{w2}\tilde{Y}_{s2}) \\ &= E\left[(\tilde{Y}_{w2}(\theta\tilde{Y}_{s1} + \omega_s'\tilde{\gamma}_s + \epsilon_{s2})\right] \\ &= \theta\sigma_{w2,s1} + \sigma_{w2,s1} \\ &= \theta\sigma_{w2,s1} + (1+\theta)\omega_w'\Gamma_{ws}\omega_s \\ &= (1+\theta)^2\omega_w'\Gamma_{ws}\omega_s. \end{aligned} \tag{4.50}$$

Next, at $k = 2$ and $t = 3$,

$$\begin{aligned}\sigma_{w2,s3} = cov(\tilde{Y}_{w2}, \tilde{Y}_{s3}) &= E(\tilde{Y}_{w2}\tilde{Y}_{s3}) \\ &= E\left[(\tilde{Y}_{w2}(\theta\tilde{Y}_{s2} + \omega_s'\tilde{\gamma}_s + \epsilon_{s3})\right] \\ &= \theta\sigma_{w2,s2} + \sigma_{w2,s1} \\ &= \theta\sigma_{w2,s2} + (1+\theta)\omega_w'\Gamma_{ws}\omega_s \\ &= (1+\theta)(1+\theta+\theta^2)\omega_w'\Gamma_{ws}\omega_s. \end{aligned} \tag{4.51}$$

In general, for $k = 2$ and $t = 1, 2, ..., m$, we conclude that

$$\begin{aligned}\sigma_{w2,st} &= cov(\tilde{Y}_{w2}, \tilde{Y}_{st}) \\ &= \theta\sigma_{w2,s(t-1)} + (1+\theta)\omega_w'\Gamma_{ws}\omega_s \\ &= (1+\theta)\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_w'\Gamma_{ws}\omega_s, \end{aligned} \tag{4.52}$$

where $\sigma_{w2,s0} = 0$. Now, for $k = 3$ and $t = 1$, we find

$$\begin{aligned}
\sigma_{w3,s1} = cov(\tilde{Y}_{w3}, \tilde{Y}_{s1}) &= E(\tilde{Y}_{w3}\tilde{Y}_{s1}) \\
&= E\left[(\tilde{Y}_{w3}(\omega_s'\tilde{\gamma}_s + \epsilon_{s1})\right] \\
&= E(\tilde{Y}_{w3}\omega_s'\tilde{\gamma}_s) \\
&= \theta E(\tilde{Y}_{w2}\omega_s'\tilde{\gamma}_s) + \omega_w'\Gamma_{ws}\omega_s \\
&= \theta\sigma_{w2,s1} + \omega_w'\Gamma_{ws}\omega_s \\
&= (1+\theta+\theta^2)\omega_w'\Gamma_{ws}\omega_s. \qquad (4.53)
\end{aligned}$$

At $k = 3$ and $t = 2$,

$$\begin{aligned}
\sigma_{w3,s2} = cov(\tilde{Y}_{w3}, \tilde{Y}_{s2}) &= E(\tilde{Y}_{w3}\tilde{Y}_{s2}) \\
&= E\left[(\tilde{Y}_{w3}(\theta\tilde{Y}_{s1} + \omega_s'\tilde{\gamma}_s + \epsilon_{s2})\right] \\
&= \theta\sigma_{w3,s1} + \sigma_{w3,s1} \\
&= (1+\theta)\sigma_{w3,s1} \\
&= (1+\theta)(1+\theta+\theta^2)\omega_w'\Gamma_{ws}\omega_s. \qquad (4.54)
\end{aligned}$$

Next, at $k = 3$ and $t = 3$,

$$\begin{aligned}
\sigma_{w3,s3} = cov(\tilde{Y}_{w3}, \tilde{Y}_{s3}) &= E(\tilde{Y}_{w3}\tilde{Y}_{s3}) \\
&= E\left[(\tilde{Y}_{w3}(\theta\tilde{Y}_{s2} + \omega_s'\tilde{\gamma}_s + \epsilon_{s3})\right] \\
&= \theta\sigma_{w3,s2} + \sigma_{w3,s1} \\
&= \theta\sigma_{w3,s2} + (1+\theta+\theta^2)\omega_w'\Gamma_{ws}\omega_s \\
&= (1+\theta+\theta^2)^2\omega_w'\Gamma_{ws}\omega_s. \qquad (4.55)
\end{aligned}$$

At $k = 3$ and $t = 4$,

$$\begin{aligned}\sigma_{w3,s4} = cov(\tilde{Y}_{w3}, \tilde{Y}_{s4}) &= E(\tilde{Y}_{w3}\tilde{Y}_{s4}) \\ &= E\left[(\tilde{Y}_{w3}(\theta\tilde{Y}_{s3} + \omega_s'\tilde{\gamma}_s + \epsilon_{s4})\right] \\ &= \theta\sigma_{w3,s3} + \sigma_{w3,s1} \\ &= \theta\sigma_{w3,s3} + (1 + \theta + \theta^2)\omega_w'\Gamma_{ws}\omega_s \\ &= (1 + \theta + \theta^2)(1 + \theta + \theta^2 + \theta^3)\omega_w'\Gamma_{ws}\omega_s. \qquad (4.56)\end{aligned}$$

In general, for $k = 3$ and $t = 1, 2, ..., m$, we conclude that

$$\begin{aligned}\sigma_{w3,st} &= cov(\tilde{Y}_{w3}, \tilde{Y}_{st}) \\ &= \theta\sigma_{w3,s(t-1)} + (1 + \theta + \theta^2)\omega_w'\Gamma_{ws}\omega_s \\ &= (1 + \theta + \theta^2)\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_w'\Gamma_{ws}\omega_s, \qquad (4.57)\end{aligned}$$

where $\sigma_{w3,s0} = 0$. Therefore from (4.48), (4.52), (4.57), and using repeated substitution method we conclude that

$$\sigma_{wk,st} = \Big\{\sum_{j=0}^{k-1}\theta^j\Big\}\Big\{\sum_{j=0}^{t-1}\theta^j\Big\}\omega_w'\Gamma_{ws}\omega_s, \qquad (4.58)$$

the recursive relation covariance is given by

$$\begin{aligned}\sigma_{wk,st} &= cov(\tilde{Y}_{wk}, \tilde{Y}_{st}) \\ &= \theta\sigma_{wk,s(t-1)} + \Big\{\sum_{j=0}^{k-1}\theta^j\Big\}\omega_w'\Gamma_{ws}\omega_s. \qquad (4.59)\end{aligned}$$

■

Now, from Lemmas (4.2.1), (4.2.2), and (4.2.3), we use Σ to denote the matrix of variance-covariance of responses as follows,

$$cov(\tilde{Y}_{wk}, \tilde{Y}_{st}) = \Sigma = \begin{cases} \Big\{ \sum_{j=0}^{t-1} \theta^j \Big\}^2 \omega_s' \Gamma_{ss} \omega_s + \Big\{ \sum_{j=0}^{t-1} \theta^{2j} \Big\} \sigma_\epsilon^2, & w = s, k = t \\ \Big\{ \sum_{j=0}^{k-1} \theta^j \Big\} \Big\{ \sum_{j=0}^{t-1} \theta^j \Big\} \omega_s' \Gamma_{ss} \omega_s + \theta^{|t-k|} \Big\{ \sum_{j=0}^{min(k,t)-1} \theta^{2j} \Big\} \sigma_\epsilon^2, & w = s, k \neq t \\ \Big\{ \sum_{j=0}^{k-1} \theta^j \Big\} \Big\{ \sum_{j=0}^{t-1} \theta^j \Big\} \omega_w' \Gamma_{ws} \omega_s, & w \neq s, k \neq t. \end{cases} \tag{4.60}$$

4.3 Estimation of the Parameters of Cluster-Based Spatial-Temporal Linear Dynamic Mixed Effect Model

Recall from previous sections that the proposed spatial-temporal response variable y_{st} (4.1) has a mean, which is a function of β, and variance and covariance, which are functions of θ, σ_ϵ^2, σ_γ^2, and ϕ. In the following subsections, we develop estimating equations for the unknown parameters of the model. As the spatial-temporal linear dynamic mixed effect model involves the estimation of five parameters, we follow Sutradhar (2011, Chapter 3) to develop a marginal GQL approach for β, σ_γ^2 and ϕ, and a marginal MM technique for σ_ϵ^2 and θ. In order to estimate each parameter, we solve the marginal estimating equation by assuming the other parameters of the model are known. In the following subsections, we discuss the marginal estimation of each parameter.

4.3.1 Marginal GQL Estimation for β

First, we rewrite the model (4.5) in matrix notations,

$$Y = \tilde{Y} + X\beta, \tag{4.61}$$

where $Y = (y_{11}, ..., y_{1m}, ..., y_{st}, ..., y_{S1}, ..., y_{Sm})'$ is the $mS \times 1$ vector of response variables, $\tilde{Y} = (\tilde{y}_{11}, ..., \tilde{y}_{1m}, ..., \tilde{y}_{st}, ..., \tilde{y}_{S1}, ..., \tilde{y}_{Sm})'$ is the $mS \times 1$ vector of mean deleted variables, $X = (x'_{11}, ..., x'_{1m}, ..., x'_{S1}, ..., x'_{Sm})'$ is the $mS \times p$ design matrix with $x_{st} = (x_{st1}, ..., x_{stp})'$ for $s = 1, 2, ..., S$ and $t = 1, 2, ..., m$. Let $\beta \in \mathbb{R}^p$ be a $p \times 1$ vector of unknown regression effects. Under the assumptions of the model and Lemma 4.2.1,

$$\mu = \mu(\beta) = E(Y) = X\beta = (\mu_{11}(\beta), ..., \mu_{1m}(\beta), ..., \mu_{st}(\beta), ..., \mu_{Sm}(\beta))', \tag{4.62}$$

where

$$\mu_{st}(\beta) = E(Y_{st}) = x'_{st}\beta, \tag{4.63}$$

and $\Sigma = cov(Y) = f(\theta, \sigma^2_\gamma, \sigma^2_\epsilon, \phi)$ such that the entries of Σ are obtained by (4.60). Clearly, $\mu(\beta)$ is a function of unknown β, and Σ is a function of the unknown scale, correlation, and dynamic parameters. In fact, by assuming the known values of θ, σ^2_γ, σ^2_ϵ, and ϕ, the GQL estimating equation for β is given by minimizing the following equation with respect to β as

$$(y - X\beta)'\Sigma^{-1}(y - X\beta). \tag{4.64}$$

Note that in this linear model because $\mu = X\beta$, the GQL estimation of β yields the GLS estimation of β as

$$\hat{\beta}_{GQL} \equiv \hat{\beta}_{GLS} = [X'\Sigma^{-1}X]^{-1}[X'\Sigma^{-1}y]. \tag{4.65}$$

4.3.2 Marginal MM Estimation for σ_ϵ^2

For the estimation of σ_ϵ^2, we use the method of moment techniques and construct a suitable unbiased estimating equation based on the basic statistics

$$W = \sum_{s=1}^{S}\sum_{t=1}^{m}(y_{st} - \mu_{st})^2/mS. \tag{4.66}$$

In the MM approach, to find an unbiased estimator for σ_ϵ^2, we should solve the following estimating equation

$$W - E(W) = 0. \tag{4.67}$$

We obtain the expectation of W by using (4.19) as follows,

$$\begin{aligned}
E(W) =&E\Big(\frac{1}{mS}\sum_{s=1}^{S}\sum_{t=1}^{m}(y_{st} - \mu_{st})^2\Big)\\
=&\frac{1}{mS}\sum_{s=1}^{S}\sum_{t=1}^{m}var(\tilde{Y}_{st})\\
=&\frac{1}{mS}\sum_{s=1}^{S}\sum_{t=1}^{m}\Big[(\frac{1-\theta^t}{1-\theta})^2\omega_s'\Gamma_{ss}\omega_s + (\frac{1-\theta^{2t}}{1-\theta^2})\sigma_\epsilon^2\Big]\\
=&\frac{1}{mS}\sum_{s=1}^{S}\omega_s'\Gamma_{ss}\omega_s\sum_{t=1}^{m}(\frac{1-\theta^t}{1-\theta})^2 + \frac{\sigma_\epsilon^2}{m}\sum_{t=1}^{m}(\frac{1-\theta^{2t}}{1-\theta^2})\\
=&\frac{\sigma_\gamma^2\big(\phi(N-S)+S\big)}{mS}\sum_{t=1}^{m}(\frac{1-\theta^t}{1-\theta})^2 + \frac{\sigma_\epsilon^2}{m}\sum_{t=1}^{m}(\frac{1-\theta^{2t}}{1-\theta^2}),
\end{aligned} \tag{4.68}$$

where $N = \sum_{s=1}^{S} n_s$. Let $\hat{\sigma}^2_{\epsilon,MM}$ be the solution of (4.67) as follows,

$$\hat{\sigma}^2_{\epsilon,MM} = m\Bigg[\sum_{t=1}^{m}\Big(\frac{1-\theta^{2t}}{1-\theta^2}\Big)\Bigg]^{-1}\Bigg[W - \frac{\sigma_\gamma^2\big(\phi(N-S)+S\big)}{mS}\sum_{t=1}^{m}\Big(\frac{1-\theta^t}{1-\theta}\Big)^2\Bigg]. \tag{4.69}$$

4.3.3 Marginal MM Estimation for θ

Following Oyet and Sutradhar (2013, Section 3.2), for the estimation of θ, we consider the standardized sample variance and the standardized sample lag one autocovariance to find a suitable estimating equation; that is,

$$S_1 = \frac{1}{mS}\sum_{s=1}^{S}\sum_{t=1}^{m}(\frac{y_{st}-\mu_{st}}{\sigma_{st}})^2, \tag{4.70}$$

$$S_2 = \frac{1}{(m-1)S}\sum_{s=1}^{S}\sum_{t=1}^{m}(\frac{y_{st}-\mu_{st}}{\sigma_{st}})(\frac{y_{s(t+1)}-\mu_{s(t+1)}}{\sigma_{s(t+1)}}), \tag{4.71}$$

with

$$E(S_1) = \frac{1}{mS}\sum_{s=1}^{S}\sum_{t=1}^{m}(\frac{E(y_{st}-\mu_{st})^2}{\sigma_{st}^2}) = 1, \tag{4.72}$$

$$\begin{aligned} E(S_2) =& \frac{1}{(m-1)S}\sum_{s=1}^{S}\sum_{t=1}^{m}\frac{E[(y_{st}-\mu_{st})(y_{s(t+1)}-\mu_{s(t+1)})]}{\sigma_{st}\sigma_{s(t+1)}} \\ =& \frac{1}{(m-1)S}\sum_{s=1}^{S}\sum_{t=1}^{m}\frac{\sigma_{st,s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}}, \end{aligned} \tag{4.73}$$

where σ_{st}(or $\sigma_{s(t+1)}$) and $\sigma_{st,s(t+1)}$ can be obtained using Lemma 4.2.1 and 4.2.2, respectively. To obtain the method of moment consistent estimator of θ, we define $r_1 = S_2/S_1$ such that

$$E(r_1) = E(\frac{S_2}{S_1}) \approx \frac{E(S_2)}{E(S_1)} = E(S_2). \tag{4.74}$$

We solve the following estimating equation for θ by assuming other parameters are known; that is,

$$g(\theta) \quad = \quad r_1 - E(r_1) = 0 \tag{4.75}$$

$$= \quad \frac{\sum_{s=1}^{S}\sum_{t=1}^{m}(\frac{\tilde{y}_{st}\tilde{y}_{s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}})/(m-1)S}{\sum_{s=1}^{S}\sum_{t=1}^{m}(\frac{\tilde{y}_{st}}{\sigma_{st}})^2/mS} - \frac{1}{(m-1)S}\sum_{s=1}^{S}\sum_{t=1}^{m}\frac{\sigma_{st,s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}} = 0. \tag{4.76}$$

The expression in (4.76) contains complex summations that make it complicated to find the equation's root for θ. For simplicity of the notation in (4.76), we set

$$g_1(\theta) \quad = \quad \frac{\tilde{y}_{st}\tilde{y}_{s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}} \tag{4.77}$$

$$g_2(\theta) \quad = \quad \frac{\tilde{y}_{st}^2}{\sigma_{st}^2} \tag{4.78}$$

$$g_3(\theta) \quad = \quad \frac{\sigma_{st,s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}}. \tag{4.79}$$

Thus, the Newton-Raphson iterative technique given by

$$\hat{\theta}_{MM}(k+1) = \hat{\theta}_{MM}(k) - \left[\left\{\partial g(\theta)/\partial\theta\right\}^{-1} g(\theta)\right]_{(k)}, \tag{4.80}$$

will be used to solve (4.76), where $[.]_{(k)}$ denotes that the expression within brackets is evaluated at $\theta = \hat{\theta}_{MM}(k)$. In (4.80), $\partial g(\theta)/\partial\theta$ is the derivative of $g(\theta)$ in (4.76) with respect to θ. In order to simplify the derivative formulas, we assume that $a = \omega_s'\Gamma_{ss}\omega_s$, and $b = \sigma_\epsilon^2$. Then,

$$\sigma_{st}^2 = \frac{(1-\theta^t)^2}{(1-\theta)^2}a + \frac{1-\theta^{2t}}{1-\theta^2}b, \tag{4.81}$$

$$\sigma_{s(t+1)}^2 = \frac{(1-\theta^{(t+1)})^2}{(1-\theta)^2}a + \frac{1-\theta^{2(t+1)}}{1-\theta^2}b, \tag{4.82}$$

$$\sigma_{st,s(t+1)} = \frac{(1-\theta^t)(1-\theta^{t+1})}{(1-\theta)^2}a + \theta\frac{1-\theta^{2t}}{1-\theta^2}b. \tag{4.83}$$

Since $g(\theta)$ is a complicated equation, we break it down into three terms and use the Maple software to compute the derivative of each term as follows:

- Derivative of $g_1(\theta) = \frac{\tilde{y}_{st}\tilde{y}_{s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}}$ with respect to θ

$$\begin{aligned}\partial g_1(\theta)/\partial\theta = &-\tilde{y}_{st}\tilde{y}_{s(t+1)}\Bigg\{\Bigg(\frac{2a(1-\theta^t)^2}{(1-\theta)^3} + \frac{2b\theta(1-\theta^{2t})}{(1-\theta^2)^2} \\ &-\frac{2at\theta^{t-1}(1-\theta^t)}{(1-\theta)^2} - \frac{2bt\theta^{2t-1}}{1-\theta^2}\Bigg)\sigma_{s(t+1)}^2 \\ &+\Bigg(\frac{2a(1-\theta^{t+1})^2}{(1-\theta)^3} + \frac{2b\theta(1-\theta^{2t+2})}{(1-\theta^2)^2} - \frac{2b(t+1)\theta^{2t+1}}{1-\theta^2} \\ &-\frac{2a(t+1)\theta^t(1-\theta^{t+1})}{(1-\theta)^2}\Bigg)\sigma_{st}^2\Bigg\}\Bigg/ 2(\sigma_{st}^2\sigma_{s(t+1)}^2)^{1.5}.\end{aligned} \tag{4.84}$$

- Derivative of $g_2(\theta) = \frac{\tilde{y}_{st}^2}{\sigma_{st}^2}$ with respect to θ

$$\begin{aligned}\partial g_2(\theta)/\partial\theta = &-\tilde{y}_{st}\Bigg\{\frac{2a(1-\theta^t)^2}{(1-\theta)^3} + \frac{2b\theta(1-\theta^{2t})}{(1-\theta^2)^2} \\ &-\frac{2at\theta^{t-1}(1-\theta^t)}{(1-\theta)^2} - \frac{2bt\theta^{2t-1}}{1-\theta^2}\Bigg\}\Bigg/(\sigma_{st}^2)^2.\end{aligned} \tag{4.85}$$

- Derivative of $g_3(\theta) = \frac{\sigma_{st,s(t+1)}}{\sigma_{st}\sigma_{s(t+1)}}$ with respect to θ

$$\begin{aligned}
\partial g_3(\theta)/\partial\theta =& \Bigg\{ \frac{2a(1-\theta^t)(1-\theta^{t+1})}{(1-\theta)^3} + \frac{b(1-\theta^{2t})}{1-\theta^2} + \frac{2b\theta^2(1-\theta^{2t})}{(1-\theta^2)^2} \\
&- \frac{at\theta^{t-1}(1-\theta^{t+1})}{(1-\theta)^2} - \frac{a(t+1)\theta^t(1-\theta^t)}{(1-\theta)^2} - \frac{2bt\theta^{2t}}{1-\theta^2} \Bigg\} \Bigg/ \sigma_{st}\sigma_{s(t+1)} \\
&- \left(\frac{a(1-\theta^t)(1-\theta^{t+1})}{(1-\theta)^2} + \frac{b\theta(1-\theta^{2t})}{1-\theta^2} \right) \Bigg\{ \Bigg(\frac{2a(1-\theta^t)^2}{(1-\theta)^3} \\
&+ \frac{2b\theta(1-\theta^{2t})}{(1-\theta^2)^2} - \frac{2at\theta^{t-1}(1-\theta^t)}{(1-\theta)^2} - \frac{2bt\theta^{2t-1}}{1-\theta^2} \Bigg) \sigma^2_{s(t+1)} \\
&+ \Bigg(\frac{2a(1-\theta^{t+1})^2}{(1-\theta)^3} + \frac{2b\theta(1-\theta^{2(t+1)})}{(1-\theta^2)^2} - \frac{2a(t+1)\theta^t(1-\theta^{t+1})}{(1-\theta)^2} \\
&- \frac{2b(t+1)\theta^{2t+1}}{1-\theta^2} \Bigg) \sigma^2_{st} \Bigg\} \Bigg/ 2(\sigma^2_{st}\sigma^2_{s(t+1)})^{1.5}. \qquad (4.86)
\end{aligned}$$

4.3.4 Marginal GQL Estimation for σ^2_γ

Under the proposed model (4.1), σ^2_γ is the variance of the location random effect. Clearly from (4.60), σ^2_γ is involved in variance components of the responses. Therefore, for the estimation of σ^2_γ, the vector of all second-order responses will be utilized. Let

$$U = (\tilde{y}^2_{11}, ..., \tilde{y}^2_{1m}, ..., \tilde{y}^2_{st}, ..., \tilde{y}^2_{S1}, ..., \tilde{y}^2_{Sm})', \tag{4.87}$$

with

$$E(U) = \lambda = (\lambda_{11}, ..., \lambda_{1m}, ..., \lambda_{st}, ..., \lambda_{S1}, ..., \lambda_{Sm})', \tag{4.88}$$

where $\lambda_{st} = E(\tilde{y}^2_{st}) = \sigma^2_{st}$, and

$$var(U) = \Omega. \tag{4.89}$$

For known β, θ, σ^2_ϵ, and ϕ, we can solve the second order GQL estimating equation for σ^2_γ as,

$$\frac{\partial \lambda'}{\partial \sigma^2_\gamma}\Omega^{-1}(U - \lambda) = 0. \tag{4.90}$$

If we assume that $\hat{\sigma}^2_{\gamma,GQL}$ denotes the solution to the marginal estimating equation in (4.90), then this solution can be obtained by the Newton-Raphson iterative method given by

$$\hat{\sigma}^2_{\gamma,GQL}(k+1) = \hat{\sigma}^2_{\gamma,GQL}(k) + \left[\frac{\partial \lambda'}{\partial \sigma^2_\gamma}\Omega^{-1}\frac{\partial \lambda}{\partial \sigma^2_\gamma}\right]^{-1}_{(k)} \left[\frac{\partial \lambda'}{\partial \sigma^2_\gamma}\Omega^{-1}(U - \lambda)\right]_{(k)}, \tag{4.91}$$

where $[.]_{(k)}$ denotes that the expression within the square bracket is evaluated using $\sigma^2_\gamma = \hat{\sigma}^2_{\gamma,GQL}(k)$ obtained at the k^{th} iteration. The formula (4.91) involves the variance-covariance matrix Ω, and the first derivative of vector λ with respect to σ^2_γ. In the following subsections, we calculate Ω and $\frac{\partial \lambda'}{\partial \sigma^2_\gamma}$.

4.3.4.1 Construction of the Covariance Matrix Ω for Estimation of σ_γ^2 Under Normality

In Lemma 4.2.1, we provided the first and second-order moments of the model (4.1). Now we use the following lemmas to help to compute the third and fourth-order moments [see also Sutradhar (2011, Section 3.3.1)].

Lemma 4.3.1. Under normality conditions, the third-order moments of mean deleted response are given by

$$E(\tilde{y}_{st}\tilde{y}_{sk}\tilde{y}_{sl}) = 0. \tag{4.92}$$

Lemma 4.3.2. Under normality conditions, the fourth-order moments of mean deleted response are given by

$$E(\tilde{y}_{wk}\tilde{y}_{st}\tilde{y}_{wl}\tilde{y}_{su}) = \sigma_{wk,st}\sigma_{wl,su} + \sigma_{wk,wl}\sigma_{st,su} + \sigma_{wk,su}\sigma_{st,wl}. \tag{4.93}$$

We write the $mS \times mS$ matrix of variance-covariance $\Omega = var(U)$ as,

$$\Omega = \begin{bmatrix} var(\tilde{Y}_{11}^2) & \cdots & cov(\tilde{Y}_{11}^2, \tilde{Y}_{st}^2) & \cdots & cov(\tilde{Y}_{11}^2, \tilde{Y}_{Sm}^2) \\ & & \vdots & \cdots & \vdots \\ & & var(\tilde{Y}_{st}^2) & & cov(\tilde{Y}_{st}^2, \tilde{Y}_{Sm}^2) \\ & & & \cdots & \vdots \\ & & & & var(\tilde{Y}_{Sm}^2) \end{bmatrix}_{mS \times mS}.$$

The construction of Ω requires to calculate $var(\tilde{Y}_{st}^2)$ and $cov(\tilde{Y}_{wk}^2, \tilde{Y}_{st}^2)$ for all values of s, t, w, and k. In the following cases, we calculate each component of this matrix using Lemmas 4.3.1 and 4.3.2.

- **Case 1:** For any value of s and t

$$\begin{aligned} var(\tilde{Y}_{st}^2) =& E(\tilde{Y}_{st}^4) - \left[E(\tilde{Y}_{st}^2)\right]^2 \\ =& 3\sigma_{st}^4 - \sigma_{st}^4 \\ =& 2\sigma_{st}^4. \end{aligned} \tag{4.94}$$

- **Case 2:** For any value of w, s and k, t

$$\begin{aligned} cov(\tilde{Y}_{wk}^2, \tilde{Y}_{st}^2) =& E(\tilde{Y}_{wk}^2 \tilde{Y}_{st}^2) - E(\tilde{Y}_{wk}^2)E(\tilde{Y}_{st}^2) \\ =& \sigma_{wk}^2\sigma_{st}^2 + 2\sigma_{wk,st}^2 - \sigma_{wk}^2\sigma_{st}^2 \\ =& 2\sigma_{wk,st}^2. \end{aligned} \tag{4.95}$$

4.3.4.2 Computation of $\frac{\partial \lambda'}{\partial \sigma_\gamma^2}$

Recall from (4.88) that the $mS \times 1$ vector λ contains $\lambda_{st} = \sigma_{st}^2$ for all values of s and t. Thus, to find $\frac{\partial \lambda'}{\partial \sigma_\gamma^2}$, we need to calculate $\frac{\partial \lambda_{st}}{\partial \sigma_\gamma^2} = \frac{\partial \sigma_{st}^2}{\partial \sigma_\gamma^2}$ for any s and t. From (4.8) and (4.10),

$$\lambda_{st} = \sigma_{st}^2 = \begin{cases} \left(\sum_{j=0}^{t-1}\theta^j\right)^2\sigma_\gamma^2 + \left(\sum_{j=0}^{t-1}\theta^{2j}\right)\sigma_\epsilon^2, & \text{for } \phi = 0 \\ \left(\sum_{j=0}^{t-1}\theta^j\right)^2\sigma_\gamma^2[n_s\phi + (1-\phi)] + \left(\sum_{j=0}^{t-1}\theta^{2j}\right)\sigma_\epsilon^2 & \text{for } \phi \neq 0. \end{cases} \tag{4.96}$$

It then follows that

$$\frac{\partial \lambda_{st}}{\partial \sigma_\gamma^2} = \frac{\partial \sigma_{st}^2}{\partial \sigma_\gamma^2} = \begin{cases} \left(\sum_{j=0}^{t-1}\theta^j\right)^2, & \text{for } \phi = 0 \\ \left(\sum_{j=0}^{t-1}\theta^j\right)^2[n_s\phi + (1-\phi)] & \text{for } \phi \neq 0. \end{cases} \tag{4.97}$$

4.3.5 Marginal GQL Estimation for ϕ

Under the proposed model (4.1), ϕ is the correlation between location random effects. Clearly from (4.60), ϕ is involved in covariance components of the responses. Therefore, for the estimation of ϕ, the vector of all lag one in a location with the same time points (location pair-wise product) second-order responses will be utilized. Let

$$V = (\tilde{y}_{11}\tilde{y}_{21}, ..., \tilde{y}_{1m}\tilde{y}_{2m}, \tilde{y}_{21}\tilde{y}_{31}, ..., \tilde{y}_{2m}\tilde{y}_{3m}, ..., \tilde{y}_{(S-1)1}\tilde{y}_{S1}, ..., \tilde{y}_{(S-1)m}\tilde{y}_{Sm})', \tag{4.98}$$

with

$$E(V) = \eta = (\eta_{11}, ..., \eta_{1m}, ..., \eta_{st}, ..., \eta_{(S-1)1}, ..., \eta_{(S-1)m})', \tag{4.99}$$

where $\eta_{st} = E(\tilde{y}_{st}\tilde{y}_{(s+1)t}) = \sigma_{st,(s+1)t}$, for $s = 1, ..., S-1$ and $t = 1, ..., m$, and

$$var(V) = \Delta. \tag{4.100}$$

For known β, θ, σ^2_ϵ, and σ^2_γ, we can solve the second-order GQL estimating equation for ϕ as,

$$\frac{\partial \eta'}{\partial \phi}\Delta^{-1}(V - \eta) = 0. \tag{4.101}$$

If we assume that the solution for marginal estimating equation in (4.101) is denoted by $\hat{\phi}_{GQL}$, then this solution can be obtained from the Newton-Raphson iterative procedure given by

$$\hat{\phi}_{GQL}(k+1) = \hat{\phi}_{GQL}(k) + \left[\frac{\partial \eta'}{\partial \phi}\Delta^{-1}\frac{\partial \eta}{\partial \phi}\right]^{-1}_{(k)} \left[\frac{\partial \eta'}{\partial \phi}\Delta^{-1}(V - \eta)\right]_{(k)}, \tag{4.102}$$

where $[.]_{(k)}$ denotes that the expression within the square bracket is evaluated using

$\phi = \hat{\phi}_{GQL}(k)$ obtained at the k^{th} iteration. The formula (4.102) involves the variance-covariance matrix Δ, and the first derivative of vector η with respect to ϕ. In the following subsections, we construct Δ and $\frac{\partial \eta'}{\partial \phi}$.

4.3.5.1 Construction of the Covariance Matrix Δ for Estimation of ϕ Under Normality

Similar to Ω, under normality conditions we use Lemma 4.3.2 to compute the elements of covariance matrix Δ. [see also Sutradhar (2011, Section 3.3.1)]. We write the $m(S-1) \times m(S-1)$ variance-covariance matrix $\Delta = var(V)$ as,

$$\Delta = \begin{bmatrix} var(\tilde{Y}_{11}\tilde{Y}_{21}) & \cdots & cov(\tilde{Y}_{11}\tilde{Y}_{21}, \tilde{Y}_{st}\tilde{Y}_{(s+1)t}) & \cdots & cov(\tilde{Y}_{11}\tilde{Y}_{21}, \tilde{Y}_{(S-1)m}\tilde{Y}_{Sm}) \\ & & \vdots & \cdots & \vdots \\ & & var(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}) & \cdots & cov(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}, \tilde{Y}_{(S-1)m}\tilde{Y}_{Sm}) \\ & & & & \vdots \\ & & & & var(\tilde{Y}_{(S-1)m}\tilde{Y}_{Sm}) \end{bmatrix}_{m(S-1)\times m(S-1)}$$

Clearly, the construction of Δ requires the calculation of $var(\tilde{Y}_{st}\tilde{Y}_{(s+1)t})$ for $s = 1, ..., S-1$ and $t = 1, ..., m$, and $cov(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}, \tilde{Y}_{wk}\tilde{Y}_{(w+1)k})$ for all values of s, t, w, and k. In the following cases, we calculate each component of this matrix using Lemmas 4.3.1 and 4.3.2.

- **Case 1:** For any value of $s = 1, ..., S-1$ and $t = 1, ..., m$

$$\begin{aligned} var(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}) =& E(\tilde{Y}^2_{st}\tilde{Y}^2_{(s+1)t}) - \left[E(\tilde{Y}_{st}\tilde{Y}_{(s+1)t})\right]^2 \\ =& \sigma^2_{st}\sigma^2_{(s+1)t} + 2\sigma^2_{st,(s+1)t} - \sigma^2_{st,(s+1)t} \\ =& \sigma^2_{st}\sigma^2_{(s+1)t} + \sigma^2_{st,(s+1)t}. \end{aligned} \quad (4.103)$$

- **Case 2:** For any value of $s = w$, and k, t

$$\begin{aligned} cov(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}, \tilde{Y}_{sk}\tilde{Y}_{(s+1)k}) =& E(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}\tilde{Y}_{sk}\tilde{Y}_{(s+1)k}) \\ & - E(\tilde{Y}_{st}\tilde{Y}_{(s+1)t})E(\tilde{Y}_{sk}\tilde{Y}_{(s+1)k}) \\ =& \sigma_{st,(s+1)t}\sigma_{sk,(s+1)k} + \sigma_{st,sk}\sigma_{(s+1)t,(s+1)k} \\ & + \sigma_{st,(s+1)k}\sigma_{sk,(s+1)t} - \sigma_{st,(s+1)t}\sigma_{sk,(s+1)k} \\ =& \sigma_{st,sk}\sigma_{(s+1)t,(s+1)k} + \sigma_{st,(s+1)k}\sigma_{sk,(s+1)t}. \end{aligned} \tag{4.104}$$

- **Case 3:** For any value of $s \neq w$, and k, t

$$\begin{aligned} cov(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}, \tilde{Y}_{wk}\tilde{Y}_{(w+1)k}) =& E(\tilde{Y}_{st}\tilde{Y}_{(s+1)t}\tilde{Y}_{wk}\tilde{Y}_{(w+1)k}) \\ & - E(\tilde{Y}_{st}\tilde{Y}_{(s+1)t})E(\tilde{Y}_{wk}\tilde{Y}_{(w+1)k}) \\ =& \sigma_{st,(s+1)t}\sigma_{wk,(w+1)k} + \sigma_{st,wk}\sigma_{(s+1)t,(w+1)k} \\ & + \sigma_{st,(w+1)k}\sigma_{(s+1)t,wk} - \sigma_{st,(s+1)t}\sigma_{wk,(w+1)k} \\ =& \sigma_{st,wk}\sigma_{(s+1)t,(w+1)k} + \sigma_{st,(w+1)k}\sigma_{(s+1)t,wk}. \end{aligned} \tag{4.105}$$

4.3.5.2 Computation of $\frac{\partial \eta'}{\partial \phi}$

Recall from (4.99) that the $m(S-1) \times 1$ vector η contains $\eta_{st} = \sigma_{st,(s+1)t}$ for all values of s and t. Thus, to find $\frac{\partial \eta'}{\partial \phi}$, we need to calculate $\frac{\partial \eta_{st}}{\partial \phi} = \frac{\partial \sigma_{st,(s+1)t}}{\partial \phi}$ for any s and t. From

(4.42) and (4.44),

$$\eta_{st} = \sigma_{st,(s+1)t} = \begin{cases} \left(\sum_{j=0}^{t-1} \theta^j\right)^2 \sigma_\gamma^2 \frac{n^*_{s,(s+1)}}{\sqrt{n_s n_{s+1}}}, & \text{for } \phi = 0 \\ \left(\sum_{j=0}^{t-1} \theta^j\right)^2 \frac{\sigma_\gamma^2}{\sqrt{n_s n_{s+1}}} \Big[n^*_{s,(s+1)} & \text{for } \phi \neq 0 \\ \quad + \phi\big(n_{s,(s+1)} + n^*_{s,(s+1)}(n^*_s + n^*_{s+1}) + n^*_{s,(s+1)}(n^*_{s,(s+1)} - 1)\big)\Big]. & \end{cases} \tag{4.106}$$

It then follows that

$$\frac{\partial \eta_{st}}{\partial \phi} = \frac{\partial \sigma_{st,(s+1)t}}{\partial \phi} = \begin{cases} 0, & \text{for } \phi = 0 \\ \left(\sum_{j=0}^{t-1} \theta^j\right)^2 \frac{\sigma_\gamma^2}{\sqrt{n_s n_{s+1}}} \Big[n_{s,(s+1)} & \text{for } \phi \neq 0 \\ \quad + n^*_{s,(s+1)}(n^*_s + n^*_{s+1}) + n^*_{s,(s+1)}(n^*_{s,(s+1)} - 1)\Big]. & \end{cases} \tag{4.107}$$

4.4 Computational Steps

Recall from (4.65) that the GQL method provides a consistent and efficient estimation for β. The MM estimators of σ_ϵ^2 and θ can be obtained from equations (4.69) and (4.80), respectively. The GQL estimators of σ_γ^2 and ϕ can also be obtained from equations (4.91) and (4.102), respectively. The following steps are performed to attain a suitable estimate for each parameter.

Step 1: Starting with an appropriate set of initial values for the scale and correlation parameters, $\xi^{(0)} = (\sigma_\epsilon^{2(0)}, \theta^{(0)}, \sigma_\gamma^{2(0)}, \phi^{(0)})'$, we construct Σ from (4.60) and then estimate β (say $\beta^{(1)}$) by using (4.65).

Step 2: With $\beta^{(1)}$ in Step 1 and selected initial values of $\theta^{(0)}$, $\sigma_\gamma^{2(0)}$, and $\phi^{(0)}$, the value of σ_ϵ^2 is updated by using (4.69) and it is called $\sigma_\epsilon^{2(1)}$.

Step 3: With $\beta^{(1)}$ in Step 1, $\sigma_\epsilon^{2(1)}$ in Step 2, and selected initial values of $\sigma_\gamma^{2(0)}$, and $\phi^{(0)}$, the value of θ is updated by using (4.80) and it is called $\theta^{(1)}$.

Step 4: With $\beta^{(1)}$, $\sigma_\epsilon^{2(1)}$, and $\theta^{(1)}$ in Steps 1, 2, and 3, respectively, and selected initial value of $\phi^{(0)}$, the value of σ_γ^2 is updated by using (4.91) and it value is called $\sigma_\gamma^{2(1)}$.

Step 5: With $\beta^{(1)}$, $\sigma_\epsilon^{2(1)}$, $\theta^{(1)}$, and $\sigma_\gamma^{2(1)}$ in Steps 1, 2, 3, and 4, respectively, the value of ϕ is updated by using (4.102) and it is called $\phi^{(1)}$. This new set of scale and correlation parameters obtained from Steps 2 through 5 is called $\xi^{(1)} = (\sigma_\epsilon^{2(1)}, \theta^{(1)}, \sigma_\gamma^{2(1)}, \phi^{(1)})'$.

Step 6: Now we use $\xi^{(1)} = (\sigma_\epsilon^{2(1)}, \theta^{(1)}, \sigma_\gamma^{2(1)}, \phi^{(1)})'$ in Step 1 to update β (say $\beta^{(2)}$). Then $\beta^{(2)}$, $\theta^{(1)}$, $\sigma_\gamma^{2(1)}$, and $\phi^{(1)}$ are used in Step 2 to improve σ_ϵ^2, and this updated value is called $\sigma_\epsilon^{2(2)}$. We iterate the above cycle until convergence is achieved up to a user-specified level of accuracy.

4.5 A Simulation Study

In this section, we investigate the behaviour of GQL estimation of β, σ_γ^2, and ϕ and moment estimation of θ and σ_ϵ^2. Several simulation studies were conducted using a sequence of $S = 100$ equi-distant locations with four time points at each location ($m = 4$). Note that similar to the familial-spatial model in Chapter 3, the distance between any two adjacent locations is equal to unit one. The user-specified distance to create a cluster of locations is assumed to be $d = 4$. Recall from (4.6) that the spatial-temporal response $\tilde{y}_{st}$ is affected by some fixed covariates, a vector of location random effects from neighbouring locations within the user-specified distance $d = 4$, and a lag one dynamic dependence parameter leading temporal correlations among the responses of different time points. That is,

$$\begin{aligned}\tilde{y}_{s1} &= \frac{1}{\sqrt{n_s}}1'_{n_s}\tilde{\gamma}_s + \epsilon_{s1}, \\ \tilde{y}_{st} &= \theta\tilde{y}_{s,t-1} + \frac{1}{\sqrt{n_s}}1'_{n_s}\tilde{\gamma}_s + \epsilon_{st} \quad \text{for } t = 2, 3, ..., m,\end{aligned} \tag{4.108}$$

with

$$\tilde{y}_{st} = y_{st} - x'_{st}\beta, \tag{4.109}$$

where $x_{st} = (x_{st1}, x_{st2}, x_{st3})'$ is a 3-dimensional fixed-covariate vector. The covariates were generated as follows:

- $x_{st1} = 1$ is an intercept covariate for $s = 1, 2, ..., S$ and $t = 1, 2, 3, 4$
- Similar to the time-dependent covariates were used by Oyet and Sutradhar (2013, Section 3.4), we define vectors of x_{st2} and x_{st3} given by

$$x_{st2} = \begin{cases} 0 & \text{for } 1 \leqslant s \leqslant S/8; \quad t = 1, 2 \\ -1 & \text{for } 1 \leqslant s \leqslant S/8; \quad t = 3, 4 \\ 1 & \text{for } S/8 + 1 \leqslant s \leqslant 3S/4; \quad t = 1 \\ 0.5 & \text{for } S/8 + 1 \leqslant s \leqslant 3S/4; \quad t = 2, 3, 4 \\ 0 & \text{for } 3S/4 + 1 \leqslant s \leqslant S; \quad t = 1, 2, 3 \\ 1 & \text{for } 3S/4 + 1 \leqslant s \leqslant S; \quad t = 4, \end{cases} \tag{4.110}$$

and

$$x_{st3} = \begin{cases} t/4 & \text{for } 1 \leqslant s \leqslant S/4; \quad t = 1,2,3,4 \\ -1 & \text{for } S/4+1 \leqslant s \leqslant 3S/4; \quad t = 1 \\ \;0 & \text{for } S/4+1 \leqslant s \leqslant 3S/4; \quad t = 2,3 \\ 0.5 & \text{for } S/4+1 \leqslant s \leqslant 3S/4; \quad t = 4 \\ (0.5+(t-1)0.5)/4 & \text{for } 3S/4+1 \leqslant s \leqslant S; \quad t = 1,2,3,4. \end{cases} \tag{4.111}$$

- For the regression coefficients, we chose two sets of $\beta = (\beta_1, \beta_2, \beta_3)'$ as,

$$\beta = (\beta_1, \beta_2, \beta_3)' \equiv (0.3, -0.5, 0.2)' \tag{4.112}$$

$$\beta = (\beta_1, \beta_2, \beta_3)' \equiv (0.8, 0, 0.1)' \tag{4.113}$$

- For the independent location random effects ($\phi = 0$)

$$\gamma_s^* \stackrel{iid}{\sim} N(0, \sigma_\gamma^2) \tag{4.114}$$

- For the correlated location random effects ($\phi \neq 0$) with pair-wise equi-correlation structure of (3.45)

$$\gamma_s^* \sim N(0, \sigma_\gamma^2) \tag{4.115}$$

- For the error model

$$\epsilon_{si} \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2) \tag{4.116}$$

For the values of the spatial correlation parameter, dynamic parameter, the variance of the location random effect, and variance of error term, we used $\phi \equiv (0, 0.3)$,

$\theta \equiv (-0.1, 0.1)$, $\sigma_\gamma^2 = 0.2$, and $\sigma_\epsilon^2 \equiv (0.25, 1.00)$, respectively.

4.5.1 Simulation Results of Cluster-Based Spatial-Temporal Linear Dynamic Mixed Effect Model

Once we generated $\tilde{y}_{st}$ (or y_{st}), we set the initial values of the parameters to $\sigma_\epsilon^{2(0)} = 0.01$, $\theta^{(0)} = 0.01$, $\sigma_\gamma^{2(0)} = 0.01$, and $\phi^{(0)} = 0.01$ and follow the computational steps in Section 4.4 to obtain estimates of the parameters up to 10^{-3} level of accuracy. The estimates of parameters are based on 300 simulations with $S = 100$ locations and $m = 4$ time points. Note that these simulation studies are very time-consuming because we are dealing with the estimation of seven parameters for equi-correlated location random effects ($\phi = 0.3$) case and six parameters for independent location random effects ($\phi = 0$). The simulated means and simulated standard errors for equi-correlated location random effects are reported in Tables 4.1 and 4.2. Similarly, the simulated means and simulated standard errors for independent location random effects are presented in Table 4.3. In both correlated and independent location random effects models, the GQL estimation of β provides satisfactory results. For instance, under equi-correlated location random effects in Table 4.1, with the true values of $\beta = (0.3, -0.5, 0.2)'$, $\sigma_\epsilon^2 = 0.25$, $\sigma_\gamma^2 = 0.2$, and $\theta = 0.1$, the GQL estimate of β obtained as $\hat{\beta} = (0.2737, -0.5037, 0.2061)'$. Also, under independent location random effects models in Table 4.3 with the same true values of parameters, the GQL estimate was $\hat{\beta} = (0.295, -0.504, 0.2044)'$.

Regarding the GQL estimate of σ_γ^2 and ϕ, Tables 4.1 and 4.2 show that both variance of location random effect and spatial correlation are bias especially when σ_ϵ^2 is large. Table 4.3 indicates that when the location random effects are independent ($\phi = 0$)), we observe an improvement in the estimation of σ_γ^2. The GQL approach indicates that when σ_ϵ^2 is small the σ_γ^2 is slightly underestimated and when σ_ϵ^2 is large the σ_γ^2 is slightly overestimated. It is noted that the MM approach produces biased estimates for θ which is not far from our expectation, especially when σ_ϵ^2 value is large. As the Newton-Raphson iterative methods are sensitive to the initial values, selecting the best initial value for θ may improve the MM estimate of this parameter.

Table 4.1: Estimate of β and $\xi = (\sigma_\gamma^2, \sigma_\epsilon^2, \phi, \theta)'$ based on 300 simulations with 100 locations and 4 time points, and with $\beta = (\beta_1, \beta_2, \beta_3)' = (0.3, -0.5, 0.2)'$, $\phi = 0.3$.

(a) Estimates of the regression parameters

σ_ϵ^2	σ_γ^2	θ	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
0.25	0.2	0.1	SM	0.2737	-0.5037	0.2061
			SSE	(0.0193)	(0.009)	(0.0007)
	0.2	-0.1	SM	0.3070	-0.4930	0.1980
			SSE	(0.0164)	(0.0090)	(0.0070)
1	0.2	0.1	SM	0.2975	-0.5207	0.1930
			SSE	(0.0205)	(0.0166)	(0.0132)
	0.2	-0.1	SM	0.2868	-0.5030	0.1970
			SSE	(0.0190)	(0.0150)	(0.0130)

(b) Estimates of the scale and correlation parameters

σ_ϵ^2	σ_γ^2	θ	Quantity	$\hat{\sigma}_\epsilon^2$	$\hat{\sigma}_\gamma^2$	$\hat{\phi}$	$\hat{\theta}$
0.25	0.2	0.1	SM	0.2618	0.1820	0.2527	0.1440
			SSE	(0.0022)	(0.0031)	(0.0127)	(0.0073)
	0.2	-0.1	SM	0.2640	0.1146	0.2270	-0.0500
			SSE	(0.0024)	(0.0035)	(0.0120)	(0.0070)
1	0.2	0.1	SM	1.0020	0.1443	0.1758	0.0628
			SSE	(0.0088)	(0.0065)	(0.0117)	(0.0063)
	0.2	-0.1	SM	1.0090	0.1472	0.1847	-0.0510
			SSE	(0.0084)	(0.0048)	(0.0980)	(0.0062)

Table 4.2: Estimate of β and $\xi = (\sigma_\gamma^2, \sigma_\epsilon^2, \phi, \theta)'$ based on 300 simulations with 100 locations and 4 time points, and with $\beta = (\beta_1, \beta_2, \beta_3)' = (0.8, 0, 0.1)'$, $\phi = 0.3$.

(a) Estimates of the regression parameters

σ_ϵ^2	σ_γ^2	θ	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
0.25	0.2	0.1	SM	0.7717	0.0070	0.0890
			SSE	(0.0189)	(0.0085)	(0.0070)
	0.2	-0.1	SM	0.7665	-0.0045	0.1100
			SSE	(0.0193)	(0.0090)	(0.0083)
1	0.2	0.1	SM	0.7816	-0.0055	0.0998
			SSE	(0.0298)	(0.0216)	(0.0230)
	0.2	-0.1	SM	0.7955	-0.0025	0.1243
			SSE	(0.0175)	(0.0147)	(0.0131)

(b) Estimates of the scale and correlation parameters

σ_ϵ^2	σ_γ^2	θ	Quantity	$\hat{\sigma}_\epsilon^2$	$\hat{\sigma}_\gamma^2$	$\hat{\phi}$	$\hat{\theta}$
0.25	0.2	0.1	SM	0.2535	0.1263	0.1449	0.1255
			SSE	(0.0023)	(0.0045)	(0.0090)	(0.0085)
	0.2	-0.1	SM	0.2434	0.1800	0.1600	-0.1122
			SSE	(0.0030)	(0.0080)	(0.0110)	(0.0100)
1	0.2	0.1	SM	0.9874	0.1412	0.1470	0.0600
			SSE	(0.0118)	(0.0068)	(0.0119)	(0.0093)
	0.2	-0.1	SM	1.0060	0.1527	0.1700	-0.0600
			SSE	(0.0090)	(0.0045)	(0.0090)	(0.0062)

Table 4.3: Estimate of β and $\xi = (\sigma_\gamma^2, \sigma_\epsilon^2, \theta)'$ based on 300 simulations with 100 locations and 4 time points, and with $\beta = (\beta_1, \beta_2, \beta_3)' = (0.3, -0.5, 0.2)'$, $\phi = 0$.

(a) Estimates of the regression parameters

σ_ϵ^2	σ_γ^2	θ	Quantity	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
0.25	0.2	0.1	SM	0.2950	-0.5040	0.2044
			SSE	(0.0120)	(0.0084)	(0.0068)
	0.2	-0.1	SM	0.2914	-0.5042	0.2024
			SSE	(0.0114)	(0.0080)	(0.0072)
1	0.2	0.1	SM	0.3018	-0.4837	0.2113
			SSE	(0.0154)	(0.0164)	(0.0127)
	0.2	-0.1	SM	0.3032	-0.4997	0.1910
			SSE	(0.1270)	(0.0140)	(0.0133)

(b) Estimates of the scale and dynamic parameters

σ_ϵ^2	σ_γ^2	θ	Quantity	$\hat{\sigma}_\epsilon^2$	$\hat{\sigma}_\gamma^2$	$\hat{\theta}$
0.25	0.2	0.1	SM	0.2370	0.1736	0.0800
			SSE	(0.0045)	(0.0080)	(0.0154)
	0.2	-0.1	SM	0.2304	0.2076	-0.1400
			SSE	(0.0051)	(0.0123)	(0.0170)
1	0.2	0.1	SM	0.8840	0.2490	0.0400
			SSE	(0.0227)	(0.0136)	(0.0122)
	0.2	-0.1	SM	0.8690	0.2367	-0.2110
			SSE	(0.0158)	(0.0247)	(0.0158)

Chapter 5

Discussion and Conclusion

Spatial data refers to data collected from different locations and spatial-temporal data refers to those data collected from different location at different time points. In this thesis we extended the cluster-based spatial linear mixed effect model of Mariathas and Sutradhar (2016) with one observation at each location to the multivariate familial-spatial and spatial-temporal linear mixed effect models. This chapter summarizes these two models and presents the simulation results. Future studies will be reviewed in the last section.

5.1 Summary and Conclusions

In the first chapter of this thesis, we reviewed the studies conducted in the fields of spatial and spatial-temporal linear models. In the spatial linear models, the main goal was constructing the correlation between responses from different locations. In all of these studies, it was assumed that there is only one observation at each location and this response was influenced by fixed covariates in addition to a one latent variable or a vector of latent variables from the same and neighboring locations. These studies have not discussed the impact of familial correlations between the observations of the same location on the responses. To fill this gap, we considered a familial-spatial model where there is a family of observations at each location. In addition to the spatial correlation between the responses of different locations, we cannot ignore the familial correlation due to a shared family random effect between the responses of the same location. Also, in the spatial-temporal setup, most studies have tended to focus on classes of separable and non-separable covariance functions. In the analysis of spatial-temporal linear models, none of these studies considered a cluster of locations. Hence, we considered the effects of the fixed covariates and the location random effects on the linear responses that follow an autocorrelation structure. Therefore, we derived the cluster-based spatial-temporal correlations between the responses of the same and different locations at the same and different time points. Regarding the familial-spatial case presented in Chapters 2 and 3, we constructed a situation where a group of observations at each location is influenced by a vector of unobservable variables arising from adjacent locations and a family random effect. Chapter 2 investigated the decomposition of two clusters and proposed the correlation structure and properties of the familial-spatial linear mixed model. Based on different user-specified distance values of d, we computed the number of locations in a cluster (n_w), the number of common locations between the two clusters (n^*_{ws}), and the number of correlated uncommon pairs of locations for two clusters (n_{ws}). We showed that the variance-covariance of responses is a function of n_w, n_s, n^*_{ws}, n_{ws}, and other unknown scale and correlation parameters of the model.

In Chapter 3, we considered the familial-spatial linear mixed effect model when the location random effects are equi-correlated or independent. We showed that the GLS estimation of regression parameters performed well under both equi-correlated and independent location random effect models. Moreover, under the independent location random effects, the MM estimations of variance components appear to achieve better estimates with small true values of σ_γ^2 and σ_α^2. The simulation studies of equi-correlated location random effects demonstrated that the MM technique yields better estimates for $\xi = (\sigma_\gamma^2, \sigma_\alpha^2, \sigma_\epsilon^2, \phi)$ when the true values of the scale parameters are smaller.

In Chapter 4, we have also expanded the cluster-based spatial random effect model of Mariathas and Sutradhar (2016) to a spatial-temporal model by considering temporal correlations between y_{st} and $y_{s,t-1}$ using dynamic dependence parameter. Through different lemmas we demonstrated that the variance-covariance of responses is a function of n_w, n_s, n_{ws}^*, n_{ws}, and other unknown scale and correlation parameters of the model. We also developed the marginal GQL estimation for β, σ_γ^2, and ϕ, and a marginal MM approach for σ_ϵ^2 and θ. As far as the estimation of regression parameters is concerned, for independent and correlated (equi-correlated) location random effects, the GQL (GLS) estimates of regression parameters showed that the estimates are very close to their corresponding true values. The GQL estimates of σ_γ^2 and ϕ perform better under the smaller values of σ_ϵ^2.

5.2 Future Works

One possible future work can be examining the performance of familial-spatial model parameters through the ML technique and comparing the ML estimates with the GLS and MM estimates. Also, a real-life data analysis can be applied to this model.

Future work should concentrate on the familial-spatial-temporal linear mixed effect model such that it is required to compute familial-spatial-temporal correlations among responses. Let y_{sit} be a response of i^{th} family member at time point t in the s^{th} location. Therefore, the proposed model for all $s = 1, ..., S$; $i = 1, ..., m$, and $t = 1, ..., T$ is given by

$$\begin{aligned} y_{si1} &= x'_{si1}\beta + \omega'_s\tilde{\gamma}_s + \alpha_{s1} + \epsilon_{si1}, \\ y_{sit} &= x'_{sit}\beta + \theta(y_{si,t-1} - x'_{si,t-1}\beta) + \omega'_s\tilde{\gamma}_s + \alpha_{st} + \epsilon_{sit} \quad \text{for } t = 2, ..., T, \end{aligned} \tag{5.1}$$

where $x_{sit} = (x_{sit1}, ..., x_{sitp})'$ is a p-dimensional environmental and individual fixed covariate vector, θ ($|\theta| < 1$) refers to an autoregressive order one dynamic dependence parameter, $\tilde{\gamma}_s = (\gamma_{s1}, ..., \gamma_{sn_s})'$ is a vector of location random effects, α_{st} is a family random effect, and ϵ_{sit} is an error term.

Another interesting extension of our spatial-temporal linear model is considering the higher-order model such as autoregressive order two AR(2) correlation structure. Let the response y_{st} at location s and time point t be related to $y_{s,t-1}$ and $y_{s,t-2}$ at time points $t-1$ and $t-2$, respectively. Therefore, the proposed model is given by

$$\begin{aligned} y_{s1} &= x'_{s1}\beta + \omega'_s\tilde{\gamma}_s + \epsilon_{s1}, \\ y_{s2} &= x'_{s2}\beta + \theta_1(y_{s1} - x'_{s1}\beta) + \omega'_s\tilde{\gamma}_s + \epsilon_{s2}, \\ y_{st} &= x'_{st}\beta + \theta_1(y_{s,t-1} - x'_{s,t-1}\beta) + \theta_2(y_{s,t-2} - x'_{s,t-2}\beta) \\ &+ \omega'_s\tilde{\gamma}_s + \epsilon_{st} \quad \text{for } t = 3, ..., m, \end{aligned} \tag{5.2}$$

where the stationary conditions are

$$\theta_2 + \theta_1 < 1, \;\; \theta_2 - \theta_1 < 1, \;\; -1 < \theta_2 < 1. \tag{5.3}$$

We remark that Mariathas (2012, Chapter 4) has constructed the marginal GQL estimating equations for β, σ_γ^2, and ϕ in the spatial setup for binary responses with correlated locations. In addition, the analysis of cluster-based familial-spatial and spatial-temporal models can be extended to categorical responses such as multinomial and binary data or count data with equally-spaced correlated locations or unequally-spaced locations design. It should be noted that modelling the correlation and constructing the estimating equations of parameters for categorical and count responses are not so easy.

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